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Topics from 20th century physics.
An introductory course for students in mathematics

APPENDIX A: MANIFOLDS

1. DEFINITION OF A MANIFOLD

1.1. *Introductory remarks*

The simplest nontrivial example of a differential manifold is a 2-dimensional surface in 3-dimensional Euclidean space. It can be obtained as the subset of points (x, y, z) of R^3, such that $F(x, y, z) = 0$, for a suitable differentiable function F. The function $F(x, y, z) = x^2 + y^2 + z^2$ gives in this manner the 2-sphere centered in the origin. Solutions of systems of $m - n$ equations $F_j(x^1, \ldots, x^m) = 0$, $j = 1, \ldots, m - n$, give differential manifolds of dimension $n < m$ as subsets of R^m. In the development of differential geometry it was at a certain point realized that it is possible and convenient to define a manifold not as a subset of a higher dimensional Euclidian space, but in a more intrinsic manner as an object on its own. In this picture an n-dimensional differentiable manifold can be intuitively visualized as a space which is smoothly patched together from open pieces of R^n.

1.2. *Topological manifolds*

In the precise discussion that follows a manifold will be defined as a *topological space* on which a *differentiable structure* is superimposed.

A *topological space* is a set X provided with a system of *open sets*. Any system of subsets of X that is closed under arbitrary unions, finite intersections and in addition contains X and the empty set $\emptyset$, can be used as system as open sets, i.e. as defining a *topology* for X. Two topological spaces X and Y are isomorphic if they are connected by a one-to-one correspondence which maps open sets in X onto open sets in Y. Such a map is called a *homeomorphism*. The notion of topological space makes it possible to speak of of limits, convergence and continuity. An *n-dimensional topological manifold* is a topological space which is locally homeomorphic with R^n, i.e. in which every point has an open neighbourhood homeomorphic with an open set in R^n. For technical reasons we include in this definition the requirement that X has the *Hausdorff property* and the property of being *second countable*. (Hausdorff: two distinct points can be separated by disjoint open sets. Second countable: each open set can be written as the union of sets from a fixed *countable* system of open sets.)

A *chart* on $\mathcal{M}$ is a pair $(\mathcal{U}, \phi)$, consisting of an open set $\mathcal{U}$ in $\mathcal{M}$ and a homeomorphism ϕ from $\mathcal{U}$ onto an open subset of R^n. Such a chart defines a system of *local coordinates*, functions $x^1, \ldots, x^n$ from U to R. An *atlas* is a collection of charts $\{U_\alpha, \phi_\alpha\}_\alpha$ such that the sets U_α cover X.

1.3. *Smooth manifolds*

Let (U_α, ϕ_α) and (U_β, ϕ_β) be two charts on $\mathcal{M}$. The map $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$ is a homeomorphism from $\phi_\alpha(U_\alpha \cap U_\beta)$, the image under ϕ_α of the overlap of U_α and U_β, onto the image of the same overlap set under ϕ_β. It is called the *transition function* associated with the two charts. This map $\phi_{\beta\alpha}$ and its inverse $\phi_{\alpha\beta}$ are continuous functions of open sets of R^n onto open sets of R^n, i.e. real-valued functions of real variables. Two charts (U_α, ϕ_α) and (U_β, ϕ_β) are called C^∞-compatible if the transition functions $\phi_{\alpha\beta}$ and $\phi_{\beta\alpha}$ are differentiable. Note that in this context ‘differentiable’ will always mean ‘infinitely differentiable’. (We will use the words ‘differentiable’, ‘infinitely differentiable’, ‘smooth’ and ‘C^∞’ interchangeably.) An atlas for which all the charts are C^∞-compatible, i.e. for which all the transition functions are differentiable, defines a C^∞- or *differentiable structure* on $\mathcal{M}$.

A *differential manifold* (*C^∞- manifold*, *smooth manifold* or just *manifold*) is a topological manifold provided with a differential structure.

A real-valued function f on $\mathcal{M}$ is differentiable (smooth, etc) iff for all the charts (U_α, ϕ_α) of the atlas which defines the differentiable structure of $\mathcal{M}$ the compositions $f \circ \phi_\alpha^{-1}$ are differentiable functions on the appropriate open sets of R^n, in the ordinary sense. A continuous map from an manifold $\mathcal{M}_1$ into a second manifold $\mathcal{M}_2$ will be called differentiable iff the corresponding maps from open sets in R^{n_1} to open sets in R^{n_2} are differentiable. If such a map has a differentiable inverse it is called a *diffeomorphism*.

2. TANGENT VECTORS AND VECTOR FIELDS

2.1. *The tangent space at a point of a manifold*

An n-dimensional manifold $\mathcal{M}$ has in every point p a *tangent space* $T_p(\mathcal{M})$. If one thinks of $\mathcal{M}$ as a submanifold of some R^m, with $n < m$, it is intuitively clear what $T_p(\mathcal{M})$ is: a first order approximation of $\mathcal{M}$ at p by an n-dimensional linear space spanned by the tangents of all curves in $\mathcal{M}$ passing through p. We make this precise in the picture in which a manifold is an intrinsic object, as defined in the preceding chapter.

Consider *curves* in $\mathcal{M}$, passing through the point p, i.e. smooth maps γ from a real interval $(-\varepsilon, +\varepsilon)$, for some $\varepsilon > 0$, into $\mathcal{M}$, such that $\gamma(0) = p$. On a coordinate neighbourhood U of p, with local coordinates $x^1, \ldots, x^n$, a curve γ is given by n smooth functions $x^1(t), \ldots, x^n(t)$. There is an equivalence relation between such curves: two curves γ_1 and γ_2 are equivalent if and only if in some system of local coordinates on has

$$\left(\frac{d}{d\tau} x_1^s(\tau)\right)_{\tau=0} = \left(\frac{d}{d\tau} x_2^s(\tau)\right)_{\tau=0},$$

for $s = 1, \ldots, n$. One checks easily that this definition is *independent* of the choice of coordinates. Curves through p cannot be added, but the equivalence classes can. Choose again local coordinates and represent two curves γ_1 and γ_2 by functions $x_1^s(\tau)$ and $x_2^s(\tau)$. Define the curve $\gamma_1 + \gamma_2$ by the functions $(x_1 + x_2)^s(\tau) = x_1^s(\tau) + x_2^s(\tau)$, for $s = 1, \ldots, n$. The definition of $\gamma_1 + \gamma_2$ depends on the choice of the local coordinates, the equivalence class $[\gamma_1 + \gamma_2]$ is however intrinsic, as one easily verifies. One similarly defines scalar multiplication on the equivalence classes. The tangent space $T_p(\mathcal{M})$ is defined as the space of these equivalence classes. An equivalence class $[\gamma]$ represents the tangent of a curve γ at p. $T_p(\mathcal{M})$ is clearly a vector space of dimension n.

2.2. *Tangent vectors as point derivations*

There is a useful alternative manner to represent elements of $T_p(\mathcal{M})$. Consider an open neighbourhood U of p. The algebra of smooth real functions on U is denoted as $C^\infty(U)$. Let $[\gamma]$ be an element of $T_p(\mathcal{M})$ and define a map $X_{[\gamma]}$ from $C^\infty(U)$ into R by the 'directional derivative'

$$X_{[\gamma]}(f) = \left(\frac{d}{d\tau} f(\gamma(\tau))\right)_{\tau=0} .$$

In local coordinates this is

$$\begin{aligned} X_{[\gamma]}(f) &= \left(\frac{d}{d\tau} f(x^1(\tau), \ldots, x^n(\tau))\right)_{\tau=0} = \\ &= \sum_{s=1}^{n} \left(\frac{\partial f}{\partial x^s}(x^1(\tau), \ldots, x^n(\tau))\right)_{\tau=0} \left(\frac{d}{d\tau} x^s(\tau)\right)_{\tau=0} , \end{aligned}$$

which shows that the definition of $X_{[\gamma]}$ indeed depends on the equivalence class. The map $X_{[\gamma]}$ has the property

$$X_{[\gamma]}(fg) = \left(X_{[\gamma]}(f)\right) g(p) + f(p)\, X_{[\gamma]}(g),$$

for all f and g in $C^\infty(U)$. A linear map from $C^\infty(U)$ into R with this property is called a *point derivation* of $C^\infty(U)$ in p. Each $[\gamma]$ in $T_p(\mathcal{M})$ gives a point derivation. One can show that each such point derivation comes from a unique equivalence class $[\gamma]$. The result is that there is a one-to-one correspondence between the set of point derivations and $T_p(\mathcal{M})$, a correspondence which is obviously a linear isomorphism with respect to the natural vector space structure of the space of point derivations.

2.3. *Vector fields*

A *vector field* on M is defined as a map which assigns in a smooth manner to each point p of M a tangent vector from $T_p(\mathcal{M})$. Smooth means here that for a local coordinate neighbourhood a vector field X is given by n smooth local functions $X^s(x^1, \ldots, x^n)$. Each tangent vector can be represented as a point derivation. This idea can be applied to vector fields; the action of the derivations at each point as maps from functions to the real numbers can be assembled to a single map from the algebra $C^\infty(\mathcal{M})$ of smooth functions into itself. In local coordinates this map acts on a locally defined function f as a first order linear differential operator as

$$X(f) = \sum_{s=1}^{n} X^s \frac{\partial f}{\partial x^s}.$$

It is a *derivation* of the algebra $C^\infty(\mathcal{M})$, i.e. it is a linear map from this algebra into itself which satisfies the relation

$$X(fg) = X(f)\, g + f\, X(g),$$

for all f and g in $C^\infty(\mathcal{M})$. A nontrivial theorem states that each derivation, i.e. each linear map from $C^\infty(\mathcal{M})$ in to itself which has this property comes from a unique vector field. This important fact allows us to see and treat vector fields as smooth assignments of tangent vectors to all points of the manifold, as first order linear differential operators, or, in a completely equivalent manner, as derivations of the algebra of smooth functions on the manifold, a simple, purely algebraic characterization that has many advantages. Vector fields can be added and multiplied by real numbers. They can also be multiplied by functions, $X \mapsto fX$, or $(fX)(g) = f(X(g))$, for all f and g from $C^\infty(\mathcal{M})$. This means that the space of vector fields is not only a real linear space but also a *module* over the algebra $C^\infty(\mathcal{M})$. (The notion of a module over an algebra or ring A is a generalization of that of a vector space over a field like R or C, with elements of A playing the role of scalars.) The composition of two derivations X and Y, $X \circ Y$, or written simply as XY, is a linear map but *not* a derivation; the commutator $[X, Y] = XY - YX$ is however again a derivation, as one easily checks. This fact can be used to define the commutator of two vector fields. The linear space of vector fields is an infinte dimensional *Lie algebra*. See for the notion of 'Lie algebra' Appendix D (*Lie groups and Lie algebras*).

2.4. *The tangent bundle*

The set of all tangent vectors for all points of the n-dimensional manifold $\mathcal{M}$ can be given in a natural way the structure of a $2n$-dimensional manifold. This manifold is called the *tangent bundle* of $\mathcal{M}$, and is denoted as $T(\mathcal{M})$. A chart (U, ϕ) on $\mathcal{M}$, with ϕ a system of local coordinates $x^1, \ldots, x^n$, gives a chart $(\hat{U}, \hat{\phi})$ on $T(\mathcal{M})$, with $\hat{U}$ the set of all tangent vectors at points of U, and $\hat{\phi}$ a system of coordinates $x^1, \ldots, x^n, y^1, \ldots, y^n$. The additional coordinates $y^1, \ldots, y^n$ are defined as follows: A tangent vector at a point p of U can be represented by a curve through p, which means in terms of the coordinates $x^1, \ldots, x^n$ by n functions $x^1(t), \ldots, x^n(t)$. The value of the coordinate y^j at p is then $y^j(p) = \left(\frac{d}{dx} x^j(t)\right)_{t=0}$. A vector field Y, which assigns a tangent vector to each point p, is in these coordinates given by n (smooth) functions $Y^j(x^1, \ldots, x^n)$.

The manifold $T(\mathcal{M})$ of tangent vectors is called tangent bundle because it is a particular example of a more general notion, that of a *vector bundle* over a manifold. A vector field is then a *section* of this bundle. This will not be discussed here.

3. COTANGENT VECTORS AND 1-FORMS

3.1. *The cotangent space at a point of a manifold*

For each point p of an n-dimensional manifold M the tangent space $T_p(\mathcal{M})$ was in the preceding chapter defined as an n-dimensional vector space 'attached' to $\mathcal{M}$ in p. The dual of $T_p(\mathcal{M})$, the n-dimensional vector space of linear maps from $T_p(\mathcal{M})$ into R, is denoted as $T_p^*(\mathcal{M})$, and is called the *cotangent space* at the point p.

3.2. 1-*forms*

One can define a 1-*form* on $\mathcal{M}$ as a map which assigns in a smooth manner to each point p of $\mathcal{M}$ a cotangent vector from $T_p^*(\mathcal{M})$. It may also conveniently be defined, without explicit reference to cotangent vectors at separate points, as a single map from $V(\mathcal{M})$, the space of vector fields, to $C^\infty(\mathcal{M})$, the algebra of smooth functions. According to this definition, a 1-form α is a map which is linear in the sense of C^∞-modules, meaning that it is a linear map $\alpha : V(\mathcal{M}) \to C^\infty(\mathcal{M})$ in the ordinary sense, with the additional property

$$\alpha(fX) = f\,\alpha(X),$$

for all X in $V(\mathcal{M})$ and f in $C^\infty(\mathcal{M})$. Multiplication of a 1-form by a function, $\alpha \mapsto f\alpha$, or $(f\alpha)(X) = f\,(\alpha(X))$, gives again a 1-form, so the space of 1-forms is a $C^\infty(\mathcal{M})$-module, in fact the *dual* of $V(\mathcal{M})$ in the sense of $C^\infty(\mathcal{M})$-modules. It is denoted as $\Omega^1(\mathcal{M})$.

3.3. *The exterior derivative*

For each function f in $C^\infty(\mathcal{M})$ we define the 1-form df as $(df)(X) = X(f)$, for each X in $V(\mathcal{M})$, with $X(f)$ the action of X as a derivation on f, as defined in the preceding chapter. This gives a map $d : C^\infty \to \Omega^1(\mathcal{M})$ which has the Leibniz property $d(fg) = (df)\,g + f\,dg$, for all f and g in $C^\infty(\mathcal{M})$ and is called the *exterior derivative*. One can show that an arbitrary 1-form can be written – nonuniquely – as a finite sum $\sum_r f_r dg_r$.

3.4. *The cotangent bundle*

The tangent bundle $T(\mathcal{M})$ was defined as set of all tangent vectors on $\mathcal{M}$ made into a manifold. We similarly define a $2n$-dimensional manifold consisting of all cotangent vectors at all points of $\mathcal{M}$. It is called the *cotangent bundle* of $\mathcal{M}$ and is denoted as $T^*(\mathcal{M})$. It is also an example of the general notion of vector bundle, with the 1-forms as sections. A chart (U, ϕ) on $\mathcal{M}$, with ϕ represented by local coordinates $x^1, \ldots, x^n$, induces a chart $(\tilde{U}, \tilde{\phi})$ on $T^*(\mathcal{M})$, with $\tilde{U}$ the set of all cotangent vectors at points of U and the map $\tilde{\phi}$ given by coordinates $x^1, \ldots, x^n, y_1, \ldots, y_n$. The additional coordinates $y_1, \ldots, y_n$ are defined as follows: A cotangent vector α_p at p is a vector from the dual $T_p^*(\mathcal{M})$ of $T_p(\mathcal{M})$ and as such a linear map from $T_p(\mathcal{M})$ to R. The value of the j^{th} coordinate y_j at p is the value of α_p on the tangent vector in $T_p(\mathcal{M})$ for which the j^{th} coordinate y^j is 1 and the others 0. A 1-form α, which assigns a cotangent vector α_p to each point p, is in these coordinates locally described by n (smooth) functions $\alpha_j(x^1, \ldots, x^n)$.

4. GENERAL DIFFERENTIAL FORMS

4.1. *Definition of a general k-form*

Consider in each point p of $\mathcal{M}$ the linear space $\wedge^k T_p^*(\mathcal{M})$, the k-fold exterior power of $T_p^*(\mathcal{M})$. This can be also defined as the space of k-linear antisymmetric maps from $T_p(\mathcal{M})$ into R. A k-*form* ω_k on $\mathcal{M}$ is a map which assigns in a smooth manner to every point p of M an element from $\wedge^k T_p^*(\mathcal{M})$. This can be seen as a single map

$$\omega_k : \underbrace{V(\mathcal{M}) \times \ldots \times V(\mathcal{M})}_{k\, times} \to C^\infty(\mathcal{M}),$$

which is k-linear and antisymmetric in the sense of $C^\infty(\mathcal{M})$-modules, i.e. it is k-linear and antisymmetric in the sense of real vector spaces and satisfies in addition

$$\omega_k(X_1, \ldots, fX_s, \ldots, X_k) = f\,\omega_k(X_1, \ldots, X_s, \ldots, X_k),$$

for all $X_1, \ldots, X_k$ in $V(M)$, all f in $C^\infty(\mathcal{M})$, and for $s = 1, \ldots, k$. k-forms can be multiplied by functions in an obvious manner; the space of k-forms is a $C^\infty(\mathcal{M})$-module, which is denoted as $\Omega^k(\mathcal{M})$. For $k > n$ the space $\Omega^k(\mathcal{M}) = (0)$, because of antisymmetry. We define the direct sum

$$\Omega(\mathcal{M}) = \Omega^0(\mathcal{M}) \oplus \Omega^1(\mathcal{M}) \oplus \ldots \oplus \Omega^n(\mathcal{M}),$$

with $\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$. The space $\Omega(\mathcal{M})$ of all forms is again a $C^\infty(\mathcal{M})$-module. It is also an (associative) *algebra*; the antisymmetry properties can be used to define a product of forms. The product of a k-form ω_k and a l-form σ_l is a $(k+l)$-form and is denoted as $\omega_k \wedge \sigma_l$. One has $\omega_k \wedge \sigma_l = (-1)^{kl}\sigma_l \wedge \omega_k$. An arbitrary k-form can be written – nonuniquely – as a finite sum of products $f_1 \wedge f_2 \wedge \ldots \wedge f_k$.

4.2. *The exterior derivative (continued)*

The exterior derivative d, introduced in 3.3 as a map from $C^\infty(\mathcal{M}) = \Omega^0(\mathcal{M})$ into $\Omega^1(\mathcal{M})$, can be extended to a map d from $\Omega(\mathcal{M})$ into itself, carrying each $\Omega^k(\mathcal{M})$ into $\Omega^{k+1}(\mathcal{M})$. It is completely determined by the requirement that it satisfies the generalized Leibniz property

$$d(\omega_k \wedge \sigma_l) = (d\omega_k) \wedge \sigma_l + (-1)^k \omega_k \wedge d\sigma_l,$$

for all k-forms ω_k and l-forms σ_l. We defined d in 3.3 on 0-forms (= functions) as

$$(df)(X) = X(f),$$

for all vector fields X, with $X(f)$ the action of X as a derivation on f. On 1-forms d is now defined as

$$(d\omega_1)(X, Y) = X(\omega_1(Y)) - Y(\omega_1(X)) - \omega_1([X, Y]),$$

and on 2-forms as

$$\begin{aligned}
&(d\omega_2)(X, Y, Z) = \\
&= X(\omega_2(Y, Z)) - Y(\omega_2(X, Z)) + Z(\omega_2(X, Y)) + \\
&-\omega_2([X, Y], Z) + \omega_2([X, Z], Y) - \omega_2([Y, Z], X),
\end{aligned}$$

for all vector fields X, Y and Z. The definition for general k-forms is

$$
\begin{aligned}
&(d\omega_k)(X_1, \ldots, X_{k+1}) = \\
&= \sum_{j=1,\ldots,\ldots,k+1} (-1)^{s+t} X_s(\omega_k(X_1, \ldots, \widehat{X_s}, \ldots, X_{k+1}) + \\
&+ \sum_{s,t=1,\ldots,k+1; s<t} \omega_k([X_s, X_t], X_1, \ldots, \widehat{X_s}, \ldots, \widehat{X_t}, \ldots, X_{k+1}),
\end{aligned}
$$

with a hat over a variable meaning that this variable is omitted. From this definition one derives the two main properties of the exterior derivative:

a. The exterior derivative is *nilpotent*, i.e.

$$d^2 = 0.$$

b. It is what is called a '*graded derivation*', i.e. it satisfies a generalized Leibniz condition

$$d(\omega_s \wedge \sigma_t) = d\omega_s \, \sigma_t + (-1)^t \, \omega_s \, d\sigma_t,$$

for all ω_s in $\Omega^s(\mathcal{M})$ and σ_t in $\Omega_t(\mathcal{M})$.

Remark: A k-form ω_k is called *closed* iff $d\omega = 0$; it is called *exact* iff there exists a $(k-1)$-form σ_{k-1} such that $\omega_k = d\sigma_{k-1}$. An exact form is closed; the converse does not need to be true.

Forms in local coordinates

Suppose that we have on an open set U of M a system of coordinates $x^1, \ldots, n^n$, with corresponding 1-forms $dx^1, \ldots, dx^n$. An arbitrary 1-form ω (or rather the restriction of such a form to U) can be written uniquely as $\omega = \sum_{j=1}^n \omega_j dx^j$, with the ω_j smooth functions in $C^\infty(U)$, written in terms of the coordinates as $\omega_j(x^1, \ldots, x^n)$. We can use the Einstein convention, a shorthand notation for this sum, and write $\omega = \sum_{j=1}^n \omega_j dx^j = \omega_j dx^j$. An arbitrary 2-form can be written – with the Einstein summation convention – as $\omega_2 = \frac{1}{2}\omega_{j_1 j_2} \, dx^{j_1} \wedge dx^{j_2}$, with a system of functions $\{\omega_{j_1 j_2}\}$ in $C^\infty(U)$, antisymmetric in the indices j_1 and j_2. The formula for a general k-form is

$$\omega_k = \sum_{j_1,\ldots,j_k=1}^n \omega_{j_1 \ldots j_k} \, dx^{j_1} \wedge \ldots \wedge dx^{j_k},$$

TO BE COMPLETED

6. RIEMANNIAN MANIFOLDS

6.1. *Introductory remarks*

The basic idea in defining the notion of an n-dimensional manifold $\mathcal{M}$ is to see it as a space which is in every point approximated by a copy of R^n. This gives first the topological and then the differential structure of $\mathcal{M}$. One can go one step further and introduce a *metric structure* by looking at R^n as an *Euclidean* space. How could one go about in defining in this spirit the distances between two nearby points of $\mathcal{M}$? Let p be a point of $\mathcal{M}$ and p' a second point close to p. The distance from p to p' should be given by a lenght along a curve γ passing through p and p' and this should be aproximately given by a length along the tangent line along γ at p, in the direction from p to p', i.e. defined on the tangent vector at p defined by γ. This leads to the idea of a metric or distance on each tangent space $T_p(\mathcal{M})$, given by a quadratic form Q_p on $T_p(\mathcal{M})$, or equivalently, by a symmetric bilinear form g_p, with $g_p(a_p, a_p) = Q_p(a_p)$, for all a_p in $T_p(\mathcal{M})$. The metrics g_p for different points p should then be smoothly assembled to a notion of a metric on the manifold itself.

6.2. *Definition of a Riemannian metric*

The intuitive ideas from the preceding section suggest a precise definition:

A *Riemannian* metric g on a manifold $\mathcal{M}$ is a smooth assignment to each point p of $\mathcal{M}$ of a positive definite symmetric bilinear form g_p on $T_p(\mathcal{M})$, i.e. a bilinear map

$$g_p(\cdot,\cdot) : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to R,$$

with the properties

$$g_p(a_p, b_p) = g(b_p, a_b),$$

for all a_p and b_p in $T_p(\mathcal{M})$, and

$$g_p(a_p, a_p) \geq 0,$$

for all a_p in $T_p(\mathcal{M})$, with $g_p(a_p, a_p) = 0$ if and only if $a_p = 0$.

It is clear that such a g is in the terminology of Chapter 5 a *covariant tensor field* of rank 2. This observation leads to an equivalent alternative definition, geometrically less intuitive but mathematically more convenient, with less emphasis on the points of $\mathcal{M}$, and more in terms of linear algebra of $C^\infty(\mathcal{M})$-modules:

A *Riemannian metric* on $\mathcal{M}$ is a symmetric $C^\infty(\mathcal{M})$-bilinear map

$$g : V(\mathcal{M}) \times V(\mathcal{M}) \to C^\infty(\mathcal{M}),$$

which is definite positive in the sense that, for each vector field X, $g(X, X)$ is a function in $C^\infty(\mathcal{M})$, which is nonnegative at each point of $\mathcal{M}$ vanishes at a point if and only if the vector field X vanishes at that point.

A Riemannian metric gives an 'infinitesimal' distance between two neighbouring points. It als gives a finite distance between arbitrary points *along a curve*: Let p_1 and p_2 be two point of $\mathcal{M}$ and γ be a curve through p_1 and p_2, with $p_1 = \gamma(\tau_1)$ and $p_2 = \gamma(\tau_2)$. The distance between p_1 and p_2 along γ is defined as the integral

TO BE CONTINUED