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Topics from 20^{th} century physics.
An introductory course for students in mathematics

APPENDIX D: LIE GROUPS AND LIE ALGEBRAS

1. GROUPS

1.1. *The definition of a group*

DEFINITION: A nonempty set G is called a *group* iff it is provided with a *multiplication*, i.e. a map

$$G \times G \to G,$$

written as

$$(g_1, g_2) \mapsto g_1 g_2,$$

which has the following properties:

1. It is *associative*, i.e. with $g_1(g_2 g_3) = (g_1 g_2) g_3$, for all g_1, g_2 and g_3 in G.
2. It has a *unit element*, i.e. an element e such that $ge = eg = g$, for all g in G.
3. Each $g \in G$ has an *inverse*, i.e. an element g^{-1} with $g^{-1} g = g g^{-1} = e$.

One shows easily that the unit element e and the inverse g^{-1} are unique.

A group G is called *abelian* or *commutative* iff $g_1 g_2 = g_2 g_1$, for all g_1 and g_2. An abelian group is often written additively, with the multiplication $g_1 g_2$ as $g_1 + g_2$, with g^{-1} as $-g$ and e as 0.

1.2. *Subgroups*

DEFINITION: A subset H of G is a *subgroup* of G if it is closed under the multiplication $(g_1, g_2) \mapsto g_1 g_2$ and contains with each g its inverse g^{-1}.

Note that this definition implies that H contains the unit element e of G, so the set H is a group in itself.

DEFINITION: A subgroup H of G is called an *invariant subgroup* or a *normal subgroup* if ghg^{-1} is in H for all h in H and all g in G.

In an abelian group every subgroup is clearly an invariant subgroup. In general non-abelian groups invariant subgroups are scarce; many groups have no (non-trivial) invariant subgroups. Such groups are called *simple*. A group that has no *abelian* invariant subgroups is called *semisimple*.

1.3. *Morphisms of groups*

DEFINITION: A map Φ from a group G_1 into a second group G_2 is called a *(group) homomorphism* if it maps the identity element of G_1 onto the identity element of G_2 and respects the multiplication, i.e. has $\Phi(gg') = \Phi(g)\Phi(g')$, for all g and g' in G_1.

Note that this definition implies that $\Phi(g^{-1}) = (\Phi(g))^{-1}$, for all g in G_1. The kernel of Φ is the subset of G_1 mapped onto the unit element of G_2. It is an invariant subgroup of G_1, as one can check easily.

A homomorphism that is both injective and surjective is called an *isomorphism*. An *automorphism* of a group G is an injective and surjective homomorphism of G onto itself. The set of automorphisms of a group G is again a group; it is usually denoted as $\mathrm{Aut}(G)$.

1.4. *Examples of groups*

Obvious examples of groups are transformation groups. Let X be a nonempty set, and $G(X)$ the set of all invertible maps of X onto itself. $G(X)$ is a group with the composition $g_1 \circ g_2$ as multiplication and the identity transformation as unit element. One can show that an arbitrary group G is isomorphic to a group of transformations of some set X. Nevertheless, one should see in this general context groups in the first place as sets with a multiplication operation, satisfying certain requirements.

Examples of finite groups:

1. The *symmetric groups* S_n, the groups of permutations of n objects. S_n has $n!$ elements and is nonabelian for $n \geq 4$.

2. The groups Z_n of *cyclic permutations*. Z_n has n elements and is abelian for all n.

There are many other more complicated finite groups. In physics one uses for example the *crystallographic groups* for a systematic description of the properties of crystalls.

For this course the important groups are *infinite groups*, in particular groups of real or complex matrices, such as for example $GL(n, R)$, the group of all invertible $n \times n$ real matrices, $O(n)$, the group of $n \times n$ orthogonal matrices and $SU(n)$, the group of unitary $n \times n$ matrices with determinant 1. As sets $GL(n, R)$, $O(n)$ and $SU(n)$ carry the structure of a (real) differentiable manifold, of dimension n^2 , respectively $\frac{1}{2}n(n-1)$ and $n^2 - 1$; such groups are called 'continuous' groups, or – more properly – *Lie groups*, and will be treated more extensively further on.

Remark: Matrix groups can of course be seen as groups of linear transformations in a finite dimensional vector space in which a basis has been chosen.

The n-dimensional *translation groups*, the spaces R^n with vector addition as group multiplication and the zero element as identity, are examples of abelian Lie groups. Elementary as these groups are, they are nevertheless of great importance in physics: R^3 as the group of translations in space, R^4 as translations in (relativistic) space-time.

1.5. *Quotient, products and semidirect products of groups*

There are various ways of constructing new groups from given groups:

1. Let G a group and H an invariant subgroup of G. Define an equivalence relation in G: $g_1 \sim g_2$ if and only if there is an element h in H such that $g_1 = hg_2$. The space of equivalence classes is denoted as G/H. It is a group with respect to the obvious multiplication $[g_1]\,[g_2] = [g_1 g_2]$, for all g_1 and g_2 in G, and with $[e]$ as unit element. G/H is called a *quotient group*. Note that if the subgroup H is not invariant, G/H is still defined as a space of equivalence classes, but is not a group.

2. Consider groups G_1 and G_2. The cartesian product $G_1 \times G_2$ has a natural group multiplication

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2'),$$

for all g_1, g_1' in G_1 and all g_2, g_2' in G_2, with (e_{G_1}, e_{G_2}) as unit element. $G_1 \times G_2$, with this group structure, is called a *(direct) product group*.

3. There is a useful generalization of the notion of direct product group. Consider two groups G_1 and G_2 together with an homomorphism Φ from G_2 into $\mathrm{Aut}(G_1)$, the group of all automorphisms of G_1. The product space $G_1 \times G_2$ can be given a Φ-dependent group structure, in general different from the direct product group structure, by defining the multiplication

$$(g_1, g_2)(g_1', g_2') = (g_1((\Phi(g_2))g_1'), g_2 g_2'),$$

for all g_1, g_1' in G_1 and g_2, g_2' in G_2. The unit element is again (e_{G_1}, e_{G_2}). The group defined in this manner is called the *semi-direct product* of G_1 and G_2, determined by Φ, and is usually denoted as $G_1 \times_\Phi G_2$. One verifies that the inverse of an element is given by the formula

$$(g_1, g_2)^{-1} = ((\Phi(g_2^{-1}))g_1^{-1}, g_2^{-1}).$$

The elements of the form (g_1, e_{G_2}), respectively (e_{G_1}, g_2), form a subgroup of $G_1 \times_\Phi G_2$ isomorphic to G_1, respectively G_2. The first subgroup is invariant; the quotient of $G_1 \times_\Phi G_2$ over it is isomorphic to G_2. One has $(g_1, g_2) = (g_1, e_{G_2})(e_{G_1}, g_2)$, so an arbitrary element of $G_1 \times_\Phi G_2$ can be written as a (unique) product of elements from both subgroups.

The semi-direct product multiplication rule is not very transparent at first sight. Its meaning can be better understood from the example of the group of *inhomogeneous linear transformations* of a vectorspace. Let x be an element of R^n, with coordinates $x_1, \ldots, x_n$, a also an element of R^n, with coordinates $a_1, \ldots, a_n$, and $T = \{T_{jk}\}$ an $n \times n$ matrix from $GL(n, R)$. An inhomogeneous linear transformations of R^n onto itself is defined as

$$x_j' = \sum_{k=1}^{n} T_{jk} x_k + a_j$$

Such transformations are clearly invertible and form a group. It is a semi-direct product $G_1 \times_\Phi G_2$, with $G_1 = R^n$, $G_2 = GL(R, n)$, and the homomorphism $\Phi : G_2 \to G_1$ given by $\Phi(T)x = Tx$, i.e. by $(Tx)_j = \sum_{k=1}^n T_{jk} x_k$. Writing such an inhomogeneous linear transformation as a pair (a, T), with a in R^n and T in $GL(n, R)$, we have as multiplication rule for two such transformations

$$(a, T)(a', T') = (a + Ta', TT').$$

The identity transformation is $(0, 1)$ and the inverse transformation is written as

$$(a, T)^{-1} = (-T^{-1}a, T^{-1}).$$

2. REPRESENTATIONS OF GROUPS

2.1. *Definition of representation*

DEFINITION: Let G be a group and V a real or complex vector space. A *(linear) representation* Π *of* G *in* V is a homomorphism Φ of G into the group $GL(V)$ of invertible linear transformations of V onto itself.

We can state this in a somewhat more explicit fashion. A representation Π of G in V assigns to every element g of G an invertible linear map $\Pi(g)$ of V onto itself such that $\Pi(e_G) = 1_V$ and $\Pi(g_1 g_2) = \Pi(g_1)\Pi(g_2)$, for all g_1 and g_2 in G. Note that this implies that $\Pi(g^{-1}) = (\Pi(g))^{-1}$.

If V is a vector space of finite dimension n, one can choose a basis. A representation Π then assigns $n \times n$ matrices to elements of G. The representation Π becomes in this manner a *matrix representation* of G.

DEFINITION: A representation Π of a group G in a complex Hilbert space is called *unitary* if the operators $\Pi(g)$ are unitary, for all g in G.

2.2. *Equivalence of representations*

A representation of a group is a homomorphism of that group onto a group of linear transformations. For two injective representations of the same group these groups of linear transformations are clearly isomorphic. They may however be different *as representations*. What it means for representations to be the same or different is expressed by the following definition:

DEFINITION: Two representations Π_1 and Π_2 of a group G, in vector spaces V_1, respectively V_2, are called *equivalent* iff there is an invertible linear map S from V_1 onto V_2 such that $\Pi_2(g) = S\Pi_1(g)S^{-1}$, for all g in G. For unitary representations the linear map S is required to be unitary.

Two representations of a group in vector spaces of different dimensions are obviously inequivalent. However, representations in vector spaces with the same dimensions may be inequivalent.

2.3. *Constructing new representations from given ones*

Let Π_1 and Π_2 be representations of the group G in vector spaces V_1, respectively V_2. There are two important ways of constructing a new representation from Π_1 and Π_2.

1. Direct sum: Consider the vector space $V_1 \oplus V_2$, the direct sum of V_1 and V_2. Define for each g in G the linear operator $(\Pi_1 \oplus \Pi_2)(g)$ in $V_1 \oplus V_2$ as

$$((\Pi_1 \oplus \Pi_2)(g))(\psi_1, \psi_2) := ((\Pi_1(g))\psi_1, (\Pi_2(g))\psi_2),$$

for all ψ_1 in V_1 and all ψ_2 in V_2. This defines the *direct sum representation* $\Pi_1 \oplus \Pi_2$.

2. Tensor product: Consider the vector space $V_1 \otimes V_2$, the tensor product of V_1 and V_2. Define for each g in G the linear operator $(\Pi_1 \otimes \Pi_2)(g)$ in $V_1 \otimes V_2$ by linear extension as

$$((\Pi_1 \otimes \Pi_2)(g))(\psi_1 \otimes \psi_2) := (\Pi_1(g))\psi_1 \otimes (\Pi_2(g))\psi_2,$$

for all ψ_1 in V_1 and all ψ_2 in V_2. This defines the *tensor product representation* $\Pi_1 \otimes \Pi_2$.

There are obvious generalizations to the notions of direct sum and tensor product representations of a finite number of given representations $\Pi_1, \ldots, \Pi_n$. There is also a technically more subtle generalization of the notion of direct sum to certain infinite sets of representations: the *direct integral*.

2.4. *Reducible and irreducible representations*

DEFINITION: A representation Π of a group G in V is called *reducible* iff there is a nontrivial linear subspace of V, i.e. not V itself and not $\{0\}$, which is invariant under $\Pi(g)$ for all g in G. The representation Π is called *irreducible* if V has no such subspace.

There is a stronger notion than reducibility:

DEFINITION: A representation Π of G in V is called *completely reducible* iff each invariant subspace has an invariant complement, i.e. if W is an invariant subspace, then there exists a second invariant subspace W' such that V can be written as the direct sum $V = W \oplus W'$.

Complete reducibility of a representation Π in a finite dimensional vector space V implies that it can be written as a direct sum representation $\Pi = \Pi_1 \oplus \ldots \oplus \Pi_p$ in $V = V_1 \oplus \ldots \oplus V_p$, with the Π_j irreducible representations of G in the invariant subspaces V_j. This reduction is unique up to equivalence.

2.5. *The central problem of group representation theory*

A central problem in the representation theory of groups is to find *all* representations of a given group, or class of groups, up to equivalence. The case of finite dimensional representations is the simplest. For many important groups all finite dimensional representations are completely reducible. In that case the irreducible representations are the building blocks of representation theory and the problem of finding all representations reduces to finding all *irreducible* representations. For large classes of groups this problem has been completely solved. The problem is more difficult for infinite dimensional representations; the notions of equivalence and irreducibility remain crucial in this case and much is known.

An important problem in the application of group theory to physics is that of finding explicitly the reduction of a given reducible representation into its irreducible components. A standard example of this problem is the reduction of a tensor product of two irreducible reprentations, which is in general reducible.

3. LIE GROUPS

3.1. *Definition of a Lie group*

DEFINITION: A group G is called an *n-dimensional Lie group* iff it is a n-dimensional (real) differentiable manifold, with the group structure such that the multiplication $(g_1, g_2) \mapsto g_1 g_2$ is a differentiable map from $G \times G$ onto G and the map $g \mapsto g^{-1}$ a differentiable map from G onto itself.

Note that this is the definition of a *real* Lie group. *Complex* Lie groups, with underlying complex manifolds will not be discussed in this course.

One can prove that a *closed* subgroup of a Lie groups is again a Lie group. The direct product $G_1 \times G_2$ of two Lie groups G_1 and G_2 is Lie group, and so is the a semidirect product if appropriate conditions are put on the homomorhpism $\Phi : G_2 \to \mathrm{Aut}(G_1)$.

A Lie group is called *compact* if the underlying manifold is compact as a topological space. The theory of compact Lie groups and their representations is much simpler than that of noncompact Lie groups, and is in many respect close to the theory of finite groups.

3.2. *Examples of Lie groups*

For applications in physics the main examples of Lie groups are, apart from the translation groups R^n, groups of real or complex matrices. Note in this respect that a translation group can also be written as a matrix group, by writing an element $a = (a_1, \ldots, a_n)$ from R^n as the $(n+1) \times (n+1)$ matrix $\{A_{j,k}\}$ with the diagonal elements equal to 1, $A_{j,n+1} = a_j$, for $j = 1, \ldots, n$, and the other matrix elements equal to 0.

Groups of $n \times n$ matrices are subsets of R^{n^2} for real matrices, or R^{2n^2} for complex matrices. Each of the following standard matrix Lie groups is defined either by an inequality or as the null set of a system of simple algebraic expressions in R^{n^2} or R^{2n^2}, and has therefore in a natural way the structure of a (real) differentiable manifold.

- The *general linear group* $GL(n, R)$, consisting of all invertible real $n \times n$ matrices. As a manifold it has clearly dimension n^2, because it is defined by restricting R^{n^2} by an inequality, n.l. the condition that the determinant of each matrix is nonzero.

- The *general linear group* $GL(n, C)$, consisting of all invertible complex $n \times n$ matrices. As a *real* manifold it has dimension $2n^2$.

- The *special linear group* $SL(n, R)$, consisting of all real $n \times n$ matrices with determinant equal to 1. It has dimension $n^2 - 1$.

- The *special linear group* $SL(n, C)$, consisting of all complex $n \times n$ matrices with determinant equal to 1. As a manifold it has (real) dimension $2(n^2 - 1)$.

In physics: $SL(2, C)$ is closely related to the Lorentz group (see below) and is needed for the description of *spin* in relativistic physics.

• The *orthogonal group* $O(n)$, consisting of the $n \times n$ orthogonal matrices. Remember that a (real) matrix is orthogonal iff its transverse is equal to its inverse. The determinant of an orthogonal matrix is equal to ± 1. As a linear transformation in R^n it leaves the positive definite quadratic form $x_1^2 + \ldots + x_n^2$ invariant. $O(n)$ has dimension $\frac{1}{2}n(n-1)$.

In physics: $O(3)$ is the group of rotations and reflections in 3-dimensional physical space.

• The *pseudo-orthogonal group* $O(m,n)$, consisting of $(m+n) \times (m+n)$ matrices that, as linear operators in R^n, leave the indefinite quadratic form $x_1^2 + x_2^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_{m+n}^2$ invariant. Its dimension is $\frac{1}{2}(m+n)(m+n-1)$.

In physics: $O(1,3)$ is the *Lorentz group*, the basic group of Special Relativity.

• The *special orthogonal group* $SO(n)$, consisting of orthogonal matrices with determinant equal to 1. Its dimension is the same as that of $O(n)$.

In physics: $SO(3)$ is the group of rotations 3-dimensional physical space and is as such the most important symmetry group in elementary quantum mechanics.

• The *unitary group* $U(n)$, consisting of all (complex) unitary $n \times n$ matrices. Remember that a matrix is unitary iff its inverse is equal to its hermitian adjoint. Its dimension as a (real) manifold is n^2.

• The *special unitary group* $SU(n)$, consisting of all unitary $n \times n$ matrices with the determinant equal to 1. It has real dimension $n^2 - 1$.

In physics: $SU(2)$ is closely related to $SO(3)$ and is needed for the description of nonrelativistic spin in quantum mechanics. $U(1)$, $SU(2)$ and $SU(3)$ play a role in elementary particle physics.

• The *inhomogenous linear groups* which can be obtained as semi-direct product from the translation groups R^n and the real matrix matrix groups, in the manner described in 1.5.

In physics: The semi-direct product of R^4 and $O(1,3)$ is called the *inhomogeneous Lorentz group* or *Poincaré group*. It is important in Special Relativity.

3.3. *Infinitesimal characterization of Lie groups*

A Lie group G can be generated from elements in a neighbourhood of the unit element e. Because G is an n-dimensional differentiable manifold, such a neighbourhood can be approximated in first order by $T_e(G)$, the tangent space of G at e. The group multiplication in G induces a bracket structure on $T_e(G)$, which makes it into an n-dimensional *Lie algebra*. This Lie algebra structe in turn characterizes the group G completely, up to a certain global topological property. A consequence of this is that for many purposes one can use the Lie algebra instead of the group itself. This may simplify matters because a Lie algebra is a linear object. Before explaining the relation between a Lie group and its associated Lie algebra, it is useful to discuss the properties of Lie algebras in a general way. This is the subject of the next chapter.

4. LIE ALGEBRAS

4.1. *Definition and basic properties*

DEFINITION: A (real or complex) vector space L is called a (real or complex) *Lie algebra* iff it is provided with a *Lie bracket*, i.e. a bilinear map

$$L \times L \to L,$$

written as

$$(u, v) \mapsto [u, v],$$

which has the following two properties:

1. It is *antisymmetric*, i.e. it has $[u, v] = -[v, u]$, for all u and v in L.
2. It satisfies the *Jacobi identity*, i.e.

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0,$$

for all u, v and w in L.

If the $[u, v] = 0$ for all u and v in L, satisfying in this trivial way the requirements for a Lie bracket, L is called an *abelian* Lie algebra.

Let L be finite dimensional Lie algebra. Choose a basis $u_1, \ldots, u_p$ in L. The Lie bracket of two basis elements, u_j and u_k, can be written as $[u_j, u_k] = \sum_{l=1}^{p} C^l_{jk} u_l$. The numbers C^l_{jk}, called *structure constants*, determine the Lie algebra L. They are of course basis dependent. It is easy to derive necessary and sufficient conditions for a set of numbers $\{C^l_{jk}\}$ to be the set of structure constants of a Lie group.

DEFINITION: A linear subspace K of the Lie algebra L is called a *Lie subalgebra* iff it is closed under the operation of taking the bracket of two elements. K is called an *invariant Lie subalgebra* or an *ideal* iff moreover $[u, v]$ is in K, for all u in K and all v in L. A Lie algebra is called *simple* iff it has no nontrivial ideals, *semisimple* iff it has no nontrivial abelian ideals.

DEFINITION: A map ϕ from a Lie algebra L_1 into a second Lie algebra L_2 is called a *(Lie algebra) homomorphism* iff it is linear and connects the brackets, i.e. satisfies $\phi([u, v]_1) = [\phi(u), \phi(v)]_2$, for all u and v in L_1.

There are obvious definitions of the notions of *isomorphism* and *automorphism*. The automorphisms of a Lie algebra L form a group denoted as $\mathrm{Aut}(L)$.

4.2. *Constructing new Lie algebras from given ones*

There are Lie algebra analogues for the group constructions discussed in 1.5:

1. For an ideal K in a Lie algebra L a *quotient Lie algebra* L/K is defined as the quotient as a vector space, provided with the obvious Lie bracket in terms of equivalence classes.

2. There is a *direct sum* $L_1 \oplus L_2$ of two Lie algebras L_1 and L_2. As a vector space it is the vector space direct sum. This is made into a Lie algebra by

defining the Lie bracket as $[(u_1, u_2), (v_1, v_2)] := ([u_1, v_1], [u_2, v_2])$, for all u_1, v_1 in L_1 and u_2, v_2 in L_2. This direct sum is sometimes called direct product and then denoted as $L_1 \times L_2$.

3. There is also a *semidirect product* of two Lie algebras L_1 and L_2, determined by the choice of an automorpism $\phi : L_2 \to \mathrm{Aut}(L_1)$. The definition of this semidirect product is omitted.

4.3. *Examples of Lie algebras*

An obvious example is the space of all linear operators in a vector space with the commutator $[S, T] = ST - TS$ as Lie bracket. It may be denoted as $gl(V)$. If V has finite dimension n, one can choose a basis in V. This leads to a matrix Lie algebra, denoted as $gl(n, R)$ for real V and $gl(n, C)$ for complex V. This notation is of course suggestive of the connection of Lie algebras with Lie groups, to be discussed further on. An example of an infinite dimensional Lie algebra is the space of vector fields on a smooth manifold, with as Lie bracket the bracket of vector fields. See for this Appendix A (*Manifolds*). Another infinite dimensional Lie algebra is the space of smooth functions on a symplectic manifold, with as Lie bracket the Poisson bracket, defined in the context of classical mechanics in Part II, 4.1.

In this course the discussion will be restricted to real finite dimensional Lie algebras. In fact, all the Lie algebras needed in our presentation of quantum theory are *matrix Lie algebras* with the usual matrix commutator as Lie bracket. We have the following representative list:

- $gl(n, R)$: all real $n \times n$ matrices; dimension n^2.
- $sl(n, R)$: the real $n \times n$ matrices with trace 0; dimension $n^2 - 1$.
- $sl(n, C)$: the complex $n \times n$ matrices with trace 0; dimension as a *real* Lie algebra $2(n^2 - 1)$.
- $o(n) = so(n)$: the real antisymmetric $n \times n$ matrices; dimension $\frac{1}{2}n(n-1)$.
- $u(n)$: the complex antihermitian $n \times n$ matrices; real dimension n^2.
- $su(n)$: the complex antihermitian $n \times n$ matrices with trace 0; real dimension $n^2 - 1$.

The notation for these Lie algebras are again suggestive for the relation with the Lie groups listed in 3.2.

4.4. *Representations of Lie algebras*

DEFINITION: Let L be a Lie algebra and V a real or complex vector space. A representation π of L in V is a Lie algebra homomorphism of L into the Lie algebra of linear operators in V, with the operator commutator as Lie bracket.

This can be stated in a more explicit fashion. A representation π of L in V assigns to every element u of L a linear map $\pi(u)$ of V into itself such that $\pi([u, v]) = [\pi(u), \pi(v)]$, for all u and v in L. The bracket on the left is the Lie bracket in L, that on the right the commutator for operators in V.

Let V be finite dimensional. With a choice of basis in V the operators $\pi(u)$ become matrices. This leads to the notion of *matrix representation* of a Lie algebra.

DEFINITION: A representation π of the Lie algebra L in the Hilbert space $\mathcal{H}$ is called *unitary* iff, for each u in L, the operator $\pi(u)$ is *antihermitian*, i.e. iff $(\pi(u)\psi_1, \psi_2) = -(\psi_1, \pi(u)\psi_2)$, for all ψ_1 and ψ_2 in $\mathcal{H}$.

We leave aside here the additional technicalities connected with unbounded operators, required by this definition in case the Hilbert space $\mathcal{H}$ is infinite dimensional. It should also be remarked that the term 'unitary' looks a bit strange at this point. It will become more meaningful after the discussion of the relation between Lie groups and Lie algebras in the next chapter.

A representation π of a finite dimensional Lie algebra L is completely determined by the representatives $\pi(u_1), \ldots, \pi(u_p)$ of a basis $u_1, \ldots, u_p$ in L, because of the fact that the homomorphism π is a *linear* map. For this reason applying the representation theory of Lie algebras is much simpler than applying that of Lie groups.

Let π_1 and π_2 be representations of the Lie algebra L in V_1, respectively V_2. The definition of the direct sum representation is the same as in the group case, n.l. as

$$((\pi_1 \oplus \pi_2)(u))(\psi_1, \psi_2) := ((\pi_1(u))\psi_1, (\pi_2(u))\psi_2),$$

for all u in L and all (ψ_1, ψ_2) in $V_1 \oplus V_2$. The definition of the tensor product representation is slightly different. It is by linear extension of

$$((\pi_1 \otimes \pi_2)(u))(\psi_1 \otimes \psi_2) := (\pi_1(u))\psi_1 \otimes \psi_2 + \psi_1 \otimes (\pi_2(u))\psi_2,$$

for all ψ_1 in V_1 and ψ_2 in V_2.

The definitions of *irreducibility*, *reducibility* and *complete reducibility* are the same as in the case of group representations.

5. THE RELATION OF LIE GROUPS TO LIE ALGEBRAS

5.1. *The Lie algebra $L(G)$ of a Lie group G*

Define, for a fixed element g_0 of G, the map $L_{g_0} : G \to G$ as $L_{g_0}g = g_0 g$. This map is called *left translation* over g_0. Because of the manifold properties of G, it is a diffeomorphism and induces therefore invertible linear maps between tangent spaces, the differentials $(dL_{g_0})_g : T_g(G) \to T_{g_0 g}(G)$. Starting from a single given tangent vector at an initial point, for which we may take the unit element e, one can obtain unique tangent vectors in all other points g by using the differentials of the left translation maps L_g, for all g in G. In this way one gets a vector field on G. This vector field is a *left invariant vector field*, i.e. it satisfies the condition $(dL_{g_2 g_1^{-1}})_{g_1} X_{g_1} = X_{g_2}$, for all g_1 and g_2 in G. All left invariant vector fields can be obtained in this manner. Restricting a left invariant vector field to e gives a tangent vector at e. This together establishes a linear correspondence between the tangent space $T_e(G)$ and the space of left invariant vector fields.

The space of all vector fields on G is an infinite dimensional Lie algebra with respect to the commutator of vector fields. One can show that the commutator of two left invariant vector field is again left invariant. This means that the left invariant vector fields form a Lie subalgebra, of finite dimension, because of the one-to-one linear correspondence with $T_e(G)$, a vector space of finite dimension. This linear correspondence induces a Lie algebra structure on $T_e(G)$.

Conclusion: The tangent space at the identity element e of an n-dimensional Lie group G has the structure of a real n dimensional Lie algebra, with a bracket induced by the commutator of left invariant vector fields on G. The tangent space $T_e(G)$ with this structure is called the Lie algebra of G. It will be denoted as $L(G)$.

5.2. *More about the relation between Lie groups and Lie algebras*

The procedure outlined in the preceding section assigns to every Lie group G a finite dimensional Lie algebra $L(G)$. According to a classical theorem this has a converse: every finite dimensional Lie algebra is the Lie algebra $L(G)$ of a Lie group G. A stronger statement holds if we restrict the global topological properties of G: every finite dimensional Lie algebra is the Lie algebra of a *unique simply connected* Lie group G.

Remember that a manifold $\mathcal{M}$ is simply connectd iff every closed curve on $\mathcal{M}$ can be contracted to a point. A connected Lie group has a *universal covering group*, i.e. a simply connected Lie group together with a surjective homomorphism from this covering group onto the given group. This covering group is in a certain sense the 'smallest' Lie group with these properties. The Lie algebra of a Lie group is isomorphic with the Lie algebra of the universal covering group.

Conclusion: There is a one-to-one correspondence between finite dimensional Lie algebras and simply connected Lie groups.

Examples:

1. The 1-dimensional translation group R^1 is simply connected; it is the universal covering group of $U(1) = SO(2)$, the group of rotations in the plane, which is connected but not simply connected. The groups R^1 and $U(1)$ obviously have the same (trivial) 1-dimensional abelian Lie algebra.

2. $O(3)$, the group of rotations and reflections in 3-dimensional Euclidean space is not connected. $SO(3)$, the group of rotations only is connected but not simply connected; $SU(2)$ is its simply connected universal covering group. The groups $O(3)$, $SO(3)$ and $SU(2)$ have isomorphic 3-dimensional Lie algebras.

Let G_1 and G_2 be Lie groups, with unit elements e_1 and e_2, and $\Phi : G_1 \to G_2$ a homomorphism. Note that in the definition of a homomorphism for Lie groups one has to require that is a diffeomorphism. It can be shown that the differential map, $(d\Phi)_{e_1} : T_{e_1}(G_1) \to T_{e_2}(G_2)$ is consistent with the Lie brackets on these tangent spaces, i.e. $(d\Phi)_{e_1}$ is a Lie algebra homomorphism.

Conclusion: A homomorphism Φ from a Lie group G_1 into a second Lie group G_2 induces a Lie algebra homomrphism from $L(G_1)$ into $L(G_2)$.

This has important consequences for representation theory. A representation Π of a Lie group G in a finite dimensional vector space V is a homomorphism of G into $GL(V)$, the group of invertible linear operators in V. It induces a homomorphism π from the Lie algebra $L(G)$ into the Lie algebra $L(GL(V))$, the Lie algebra of all linear operators in V, i.e. it induces a representation π of $L(G)$ in V. It will become clear in the next section that the original group representation Π can in general be recovered by exponentiation from the Lie algebra π. This means that the theory – and practice! – of representing Lie groups, can to a large extent be reduced to representing Lie algebras, a reduction of analysis and geometry to pure algebra.

5.3. *Matrix Lie groups and matrix Lie algebras*

An element a of the Lie algebra $L(G)$ is a tangent vector of G, at the unit element e. Tangent vectors at a point can be seen as equivalence classes of smooth curves $\gamma(\tau)$ through that point, with the same derivative at $\tau = 0$. One can show that for each a in $T_e(G)$ there is among the curves which have a as their tangent vector a *unique* curve $\gamma(\tau)$, defined for all τ in R^1, with the property $\gamma(\tau_1 + \tau_2) = \gamma(\tau_1)\gamma(\tau_2)$, for all real τ_1 and τ_2, i.e. such that $\gamma(\tau)$ is a 1-parameter subgroup of G. This curve is *symbolically* written as $\gamma(\tau) = e^{\tau a}$.

In the case in which G is a group of $n \times n$ matrices, γ is a smooth matrix-valued function, which can be differentiated at $\tau = 0$, giving $\left(\frac{d}{d\tau}\gamma(\tau)\right)_{\tau=0} = a$, with a an $n \times n$ matrix, in general not belonging to G, but of a different type. The formula $\gamma(\tau) = e^{\tau a}$ is no longer symbolic; it is a matrix formula with on the right hand side a matrix exponential function, well-defined for an arbitrary $n \times n$ matrix. The matrices a obtained in this manner form in fact the Lie algebra $L(G)$, as a matrix Lie algebra, with the matrix commutator as Lie bracket.

With this in mind one may look at the examples of matrix Lie algebras given in Section 3.2:

• Take $\gamma(\tau)$ in $GL(n, R)$ or $GL(n, C)$. Differentiation in $\tau = 0$ gives a general $n \times n$ real or complex matrix. So the corresponding Lie algebras are the Lie algebras of all $n \times n$ real or complex matrices, denoted as $gl(n, R)$, respectively $gl(n, C)$.

• For a finite dimensional matrix b one has the well-known formula $\det e^b = e^{\operatorname{tr} b}$. Using this one finds easily that differentiating a curve $\gamma(\tau) = e^{\tau a}$ in $SL(n, R)$ at $\tau = 0$ gives an $n \times n$ matrix with trace 0. So $L(SL(n, R)) = sl(n, R)$, the Lie algebra of all $n \times n$ matrices with trace 0. Similarly one has $L(SL(n, C)) = sl(n, C)$.

• An orthogonal matrix O has the property $O^T O = O O^T = 1$, with O^T the transverse of O. For a curve $\gamma(\tau) = e^{\tau a}$ with values in $O(n)$, differentiating the left hand and the right hand side of the equation $(\gamma(\tau))^T \gamma(\tau) = 1$ at $\tau = 0$ gives immediately $a^T + a = 0$, i.e. a is antisymmetric. So $L(O(n))$ is the algebra of allantisymmetric $n \times n$ matrices, which was denoted as $o(n)$ in Section 3.2. Note that a smooth curve going through the identity element of $O(n)$ has only values in the connected part of $O(n)$ which is $SO(n)$. Moreover an antisymmetric matrix has trace 0, so $so(n) = o(n)$.

• Using the identity $U^*U = UU^* = 1$ for unitary matrices gives in a similar way $a^* + a = 0$, for the matrix a in a curve $\gamma(\tau) = e^{\tau a}$ in $U(n)$, so $L(U(n))$ is the Lie algebra of antihermitian matrices, which was denoted as $u(n)$ in 3.2. An antihermitian matrix does in general not have trace 0, so $su(n) \neq u(n)$.

The remark about the importance of the relation between a representation Π of a Lie group G and the induced representation π of the Lie algebra $L(G)$ made at the end of 3.2 can be expanded and made more explicit for the case in which G is a matrix Lie algebra, and therefore $L(G)$ a matrix Lie algebra. For a representation in a finite dimensional vector space V one has the important and very useful relation

$$\Pi(\gamma(\tau)) = \Pi(e^{\tau a}) = e^{\tau\pi(a)}.$$

This shows us in a very explicit way how the problem of finding representations of a Lie group can be reduced to the purely algebraic problem of finding the representations of the corresponding Lie algebra.