Quark matter influence on observational properties of compact stars

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Master's thesis Theoretical Physics & Astronomy and Astrophysics Supervisors: Dr. Daniel Boer Dr. Rudy Wijnands

Three quarks for Muster Mark!
Sure he hasn't got much of a bark
And sure any he has it's all beside the mark.
But O, Wreneagle Almighty, wouldn't un be a sky of a lark
To see that old buzzard whooping about for uns shirt in the dark
And he hunting round for uns speckled trousers around by Palmerstown Park?
Hohohoho, moulty Mark!
You're the rummest old rooster ever flopped out of a Noah's ark
And you think you're cock of the wark.
Fowls, up! Tristy's the spry young spark
That'll tread her and wed her and bed her and red her
Without ever winking the tail of a feather
And that's how that chap's going to make his money and mark!

From: Finnegans Wake James Joyce



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Abstract

Observations of compact stars show again and again that the theory describing these objects is far from complete. In this thesis the possibility that free quark matter is present in such objects will be considered, studying the observational consequences of quark matter in compact stars. Free quark matter can be present at high densities as the coupling constant of the strong force becomes weaker at high energies. At low energies quarks are confined into hadrons, but at high energies the coupling constant might become so weak that deconfinement occurs. Compact stars thereby form a very interesting laboratory to study matter that cannot be made on earth.

One of the main differences between quark matter and hadronic matter is its different relation of pressure and energy, expressed in the equations of state. Compact stars made of quark matter are smaller, and the mass-radius relation shows an increasing mass with radius for most of the domain of allowed masses. This in sharp contrast with hadronic matter where the mass-radius relation has a negative slope for the entire mass domain. Furthermore, magnetic fields behave differently in quark matter and hadronic matter. Where hadronic matter is a superconductor, quark matter behaves only as a very good conductor. This has important influences on changes in the electromagnetic field, for example when the compact star is precessing. We will see that study of the behaviour of the magnetic field provides us with a very good probe of the interior of a compact star.

Preface

This thesis concludes of one and a half year of research on observational properties of quark stars, based on quantum field theoretical calculations of quark matter. I have done my research at the 'Anton Pannekoek Institute for Astronomy' of the University of Amsterdam and at the department of Theoretical Physics of the Vrije Universiteit, as my Master's research the combined Masters for Theoretical Physics and Astronomy and Astrophysics.

The theory of colour interactions (QCD) has the peculiar behaviour that the interaction strength declines for higher energies. Therefore, calculations at high energies are much less challenging than calculations at low energies. At the end of the seventies Baluni (1978), Freedman and McLerran (1977c) already calculated the QCD thermodynamic potential at high densities. Their results lead to the suggestion by Witten (1984) that in the early universe densities may have been such that quark stars have emerged. Soon after it was realised that the very high temperatures in the early universe would have quickly destroyed these stars (Alcock and Farhi, 1985).

The idea that at high densities quark behave as weakly interacting particles, however, has since been considered a real possibility for the dense objects remaining after the death of a massive star. These objects are generally known as neutron stars, although they might contain quark matter. I will use the term compact star for these objects, in contradiction with the usual definition of compact star which also includes white dwarf stars and black holes. Usually compact objects as defined here are referred to as neutron stars, a very unfortunate name that can lead to ambiguities. I will reserve the term 'neutron star' explicitly for stars made of only baryonic matter. The term 'quark star' will be used only for compact stars made of deconfined quark matter. For objects known to contain both I will use the term 'hybrid star'.

I will first discuss the observational properties of these objects and discuss the formation and evolution in chapter 1. Followed by a discussion of the equations governing compact star structure in chapter 2. In chapter 3 I will discuss some concepts quantum field theory and more specifically quantum chromodynamics. Using these concepts, I will show some details of the derivation of the quark matter thermodynamic potential in chapter 4. Based on this thermodynamic potential, I have performed calculations of stellar models. My results can be found in 5.

One particular property of quark matter is colour superconductivity. Although of little influence on the mass-radius relations, it has profound influence on the behaviour of magnetic fields. The theory of both superconductivity and colour superconductivity is shortly reviewed in chapter 6. Based on this, I will discuss magnetic field behaviour and its observational consequences for both neutron stars and quark stars in chapter 7. All conclusions are summarised in chapter 8.

Sjoerd Hardeman Amsterdam, March 30, 2007

Contents

Abstract i						
Preface						
1	Intro	oduction	1			
	1.1	Compact star observations	1			
		1.1.1 Pulsars	1			
		1.1.2 X-ray binaries	4			
		1.1.3 Surface radiation of compact stars	6			
		1.1.4 Free precession	6			
	1.2	Formation and evolution of compact stars	7			
		1.2.1 Stellar evolution and the formation of compact stars	7			
		1.2.2 Supernovae	8			
		1.2.3 Magnetic fields	10			
		1.2.4 Cooling of compact stars	12			
2	Stru	icture of compact stars	15			
	2.1	General structure equations	15			
		2.1.1 Polytropes	16			
	2.2	Modelling stars	18			
		2.2.1 Models of neutron stars and quark stars	18			
		2.2.2 Solving the models	19			
		2.2.3 Equations of state	20			
		2.2.4 Effect of rotation on compact star structure	21			
3	Qua	ntum chromodynamics at high densities	23			
	3.1	Symmetry groups	23			
	3.2	Quantum field theory	24			
		3.2.1 Instantons	26			
	3.3	Confinement and asymptotic freedom	27			
		3.3.1 Quantum chromodynamics at low energies	28			
		3.3.2 Phase transitions in dense matter	29			
	3.4	The running coupling constant in QCD	30			
		3.4.1 The running of the coupling constant	31			
		3.4.2 The running of the quantum chromodynamics coupling constant	31			

CONTENTS

4	The	rmodynamics of degenerate quark matter	33
	4.1	Fermi-Dirac distribution	33
	4.2	The MIT bag model	35
	4.3	Finite temperature field theory	35
		4.3.1 The thermodynamic potential	37
		4.3.2 Thermodynamic quantities of a quark gas	39
5	Mas	s-radius relation of quark stars	43
	5.1	Numerical stellar models	43
		5.1.1 The MIT bag model	44
	5.2	Numerically solving models of quark stars	46
		5.2.1 The method	46
		5.2.2 Equations of state used in the model	47
		5.2.3 The programme	48
	5.3	The equation of state from perturbative quantum chromodynamics	48
		5.3.1 Equation of state of three massless quarks	48
		5.3.2 Equation of state for one massive quark flavour	50
	5.4	Numerical solutions to quark star models using a perturbative QCD equation of state	53
		5.4.1 Quark star models using an equation of state of three massless quarks	53
		5.4.2 Quark star models using an equation of state of one massive quark flavour	55
	5.5	Conclusions on the mass-radius relation	56
6	The	ory of superconductivity	59
	6.1	Superconductivity in neutron matter	59
		6.1.1 Type I superconductivity	60
		6.1.2 Type II superconductivity	61
	6.2	Colour superconductivity	63
		6.2.1 Colour-Flavour Locked phase	64
		6.2.2 Two-quark superconducting phase	64
		6.2.3 One flavour pairing	65
		6.2.4 Other colour superconducting phases and the quantum chromodynamics phase	
		diagram	65
7	Mag	metic fields in hybrid stars and quark stars	67
	7.1	Neutron star magnetic behaviour	67
		7.1.1 Fluxoid pinning	67
		7.1.2 Quark matter influence on magnetic fields	70
	7.2	Consequences of quark matter for pulsar timing observations	72
		7.2.1 Glitches	72
		7.2.2 Precession	72
8	Con	clusions and discussion	75
	8.1	Mass-radius relations	75
	8.2	Quark matter and superconductivity	77
A	Deri	vation of the TOV equation	79
В	Nota	ation	83

Acknowledgements	85
Samenvatting	87

CHAPTER 1

Introduction

In this chapter, some of the general properties of compact stars will be discussed. Throughout this thesis, I will use the term compact star for an object that can either be a quark star or a neutron star. If the term 'neutron star' or 'quark star' is used, I intent to discuss objects with a known matter content. This is different from the usual definitions. Compact stars normally includes black holes and white dwarfs, while the term 'neutron star' is usually used for all objects with radii ~ 10 km, masses ~ M_{\odot} and are not black holes.

In the first part of this chapter the observational properties of compact stars are discussed. The second part covers the theory of compact star formation and evolution to explain some of the observations.

1.1 Compact star observations

First we will consider some observational features of compact stars which can be used to distinguish between different neutron star and quark star models. Compact stars are most often observed as pulsars, spinning stars with strong magnetic fields. Due to the spinning field pulsars produce very strong radiation, and can therefore easily be observed.

Even more interesting in this sense are the pulsar-white dwarf binaries and the double compact stars, of which one of the stars is a pulsar. The radiation produced by pulsars is closely related to the spin and therefore almost exactly periodic. When altered by general relativistic effects, these objects allow for very accurate measurements.

Other mechanisms of producing electromagnetic radiation are found in binary systems of a normal star with a compact star companion. If mass transfer occurs powerful electromagnetic radiation is produced. Using that these objects reside in binary systems masses can be determined with reasonable accuracy.

Finally, in Rosat surveys, isolated compact stars have been found (Agüeros et al., 2006) by their X-ray emission. These objects produce no observable pulsar radiation.

1.1.1 Pulsars

The first pulsars were discovered by Jocelyn Bell-Burnell and Anthony Hewish (Hewish et al., 1968, Pilkington et al., 1968). Pulsars are objects producing highly periodic pulses of electromagnetic radiation (hence the name: pulsating star). Very soon it was realised that these objects were associated with rotating compact stars. Strong evidence for this conclusion came with the discovery of the Vela and

Chapter 1: Introduction

Crab pulsars (see figure 1.1), associated with recent supernovae (see section 1.2.2). In Glendenning (2000, ch. 5) a historical overview is given.

When a star collapses, magnetic flux preservation predicts a strong increase in magnetic fields. Also, angular momentum conservation predicts high spin rates. A spinning compact star with a strong magnetic field will produce very strong low frequency electromagnetic radiation. Via a not fully understood mechanism, this leads to the production of observable electromagnetic radiation from the magnetic poles of the compact star.

In general, the magnetic poles and geographic poles do not coincide. As radiation is emitted in beams from the magnetic poles, these beams will sweep around since these poles are spinning with the compact star. If you happen to be in the path of such a beam, you will observe periodical emission from this beam, with the spin frequency of the compact star as the period.

The emission of the electromagnetic waves requires huge amounts of energy. The energy radiated this way is drawn from the spin frequency, as "dragging" the magnetic field through the vacuum provides a resistance and slows the star down. Using this model, magnetic fields can be calculated from spindown rates, the rate of change of the compact star spin. Assuming a dipole field the spindown rate is given by (Bhattacharya and Srinivasan, 1995)

$$B = \sqrt{\frac{3I}{8\pi^2 R^6 \sin^2 \alpha} P \dot{P}}$$
(1.1)

This formula can be used to draw lines of equal field strength in the $P - \dot{P}$ diagram (figure 1.3). For normal pulsars this suggests field strengths $\propto 10^8$ T. These magnetic fields are of the order of the expected magnetic field in case of flux preservation as will be discussed in section 1.2.3, with the exception of magnetars and millisecond pulsars.

The energy of the radiation has to come from its angular momentum. It cannot come from a decay of the electromagnetic field. Besides the theoretical reasons provided hereafter, the energetics of pulsar radiation match the energy drawn from rotation very well (Glendenning, 2000, and references therein).

Neutron stars are thought to possess in their core a superconducting proton fluid, and an almost perfect conducting electron fluid. Superfluids do not allow magnetic fields to penetrate. This is known as the Meissner effect. Depending on the type of superconductivity, the magnetic







Figure 1.2: Spindown rate the Vela pulsar. Just after a glitch the spindown rate is higher than normal (image from Lyne and Graham-Smith, 2005)

field will form fluxoids or non-superconducting domains to penetrate the superconductor. In order to lower the magnetic field of such a system, these fluxoids and domains have to move to the surface of

the superconductor in order for the field lines to reconnect and decay. However, a moving magnetic field will produce an electric field. The electron fluid then generates an opposed field preventing the vortex or domain from moving. This mechanism ensures very stable magnetic fields. It is discussed in much more detail in chapter 7.

One important observational feature of young pulsars are the glitches. As described above, pulsars spin down due to the breaking effect of the pulsar mechanism. However, sometimes, for very short periods, the star actually spins up. After this sudden change in spin the star relaxes exponentially to a new spin-down rate, as shown in figure 1.2.

Millisecond pulsars



Figure 1.3: Diagram of period versus spindown rate. Indicated are the field strength lines in Gauss $(10^4 G = 1 T)$. Points marked by a circle are binary systems, Stars represent magnetars (see section 1.1.1) The line marked by 10^{10} yr indicates the Hubble time, the spin up line indicates how a normal pulsar should evolve to become a millisecond pulsar (image from Manchester, 2006)

Millisecond pulsars are fast spinning pulsars with a low magnetic field $\propto 10^4$ T. These pulsars are thought to be created from normal pulsars in binary systems, spun up by matter accreted from their companion stars (see section 1.1.2). This model is supported by the fact that most of the millisecond pulsars are found in binary systems with a white dwarf companion. Up to today, there are no binary pulsar systems in which one of the two companions is a millisecond pulsar (Thorsett and Chakrabarty, 1999). This supports the idea that the millisecond pulsar is spun up due to mass transfer. In systems with a more massive companion, the faster evolution of the companion will leave less time for spinning up the compact star (Bhattacharva and Srinivasan, 1995). There are also a few millisecond pulsars which do not have a companion. It is thought that in these so-called black widow systems the pulsar beam passes over the location of the original companion. Calculations show that it is possible that this beam evaporates the companion star (Bhattacharya and Srinivasan, 1995). Observational support is found by Wolszczan and Frail (1992), who observed a planetary system in PSR1257+12, made from the debris of evaporation.

During the accretion, a mechanism for reducing the magnetic field must have been at work. In Mandal et al. (2006) an explanation for this decay is sought in the presence of quark

matter. They suggest the minimum observed fields $\propto 10^4$ T can be explained assuming a quark star in a superconducting phase. However, their suggestion is in contrast with the result of Alford et al. (2000) that colour superconducting quark matter is not electromagnetically superconducting. Another explanation can be provided by a locking of magnetic field lines to rotational vortices, as is expected to happen in some superconductors. These results will be explored further in chapters 6 and 7. In binary systems spin periods can be severely altered, with the longest period observed at almost 1000 seconds (Bhattacharya and Srinivasan, 1995). In such a spindown a large portion of the magnetic field might be driven out. In contrast to this result observations show that in the slowly spinning pulsars in binary systems with a massive companion, the high mass X-ray binaries, the fields are still ~ 10^8 T, as in ordinary pulsars (Coburn et al., 2006).

Magnetars

Magnetars are compact stars with a very strong magnetic field $\propto 10^{10}$ T. The stars are characterised by their high spin down rates and X-ray emission. It is impossible to explain the emission from these systems by rotation energy loss only. It is thought that the decay of the electromagnetic field also contributes, and dominates the spindown emission (Kouveliotou et al., 1998). Magnetars are marked by stars in the upper-right of the $P - \dot{P}$ diagram (figure 1.3).

Magnetic fields of magnetars are stronger than the critical field above which baryon matter becomes non-superconducting. Ohmic dissipation, the conversion of electric currents to heat through Ohmic resistance, can therefore be a mechanism for field decay and energy generation. As discussed in chapter 6, normally conducting nuclear matter has an extremely high electron conductivity. This would lead to very long decay times for the magnetic fields, making Ohmic dissipation not a very efficient mechanism for field decay.

Magnetars are often associated with quark matter. In Niebergal et al. (2006) quark superconductivity is used to explain the large magnetic fields. However, as discussed in section 6.2 and in Alford et al. (2000) the main colour superconducting phases are not superconductors with respect to electromagnetic fields. Therefore I consider this explanation unlikely.

Magnetars sometimes present giant X-ray outbursts. So far three have been observed (SGR 0526-66, SGR 1806-20 and SGR 1900+14). Israel et al. (2005) has discovered quasi periodic signals in these outbursts, that were attributed to star quakes (Strohmayer and Watts, 2005, Watts and Strohmayer, 2006). This model only works for a crust thickness too large for a pure quark star. The authors therefore ruled out that magnetars are quark stars. In contradiction to this, Levin (2006) used a toy model to show that the model used suggests very strong damping of the high frequencies. As these frequencies have been observed, the author considers the star quake model unlikely. His suggestion is that the quasi periodic signals are associated with wave phenomena in the magnetar magnetic field.

1.1.2 X-ray binaries

Many compact stars have been found in binary systems in which mass transfer occurs. Due to the very strong gravitational field of a compact star, up to 30 percent of the rest energy of the infalling mass can be converted to energy. This leads to very high temperatures and causes these objects to be mainly visible in X-rays. This allows for easy observation of compact stars in these binaries.

The binary has a very strong influence on the compact star. First, stars in a binary system will evolve very different from a solitary star, as mass transfer can seriously alter the initial masses of stars. This is demonstrated in the Algol system, in which the lighter star has evolved much further, while it is commonly accepted that the lifetime of a star is inversely related to its initial mass. Based on the mass-luminosity relationship $L \propto M_0^{3.5}$ (Close et al., 2005) with L the luminosity and M_0 the initial mass one can obtain $T_{\text{max}} \propto M_0^{-2.5}$ for the lifetime T_{max} . For the last step the assumption is made that the available fuel of a star is proportional to its mass by $LT_{\text{max}} \propto M$. The explanation for this system is that initially the lighter star was more massive than its companion, but that accretion has changed this situation dramatically.

Accretion

Accretion occurs when mass from a normal star is transported via some mechanism to the compact star. There are two different mechanisms: wind accretion and Roche Lobe overflow.



Figure 1.4: *The potential of two rotating masses with a mass ratio of 2:1*

Roche lobe overflow occurs when the companion star grows larger than its Roche lobe, the maximum equipotential surface in which the gravity of the star dominates. This can be understood from a corotating potential

$$\phi = -\frac{GM_1}{\mathbf{r} - \mathbf{r}_1} - \frac{GM_2}{\mathbf{r} - \mathbf{r}_2} - \frac{1}{2}(\mathbf{\Omega} \times \mathbf{r})^2 \qquad (1.2)$$

with M_1 and M_2 the masses of the objects and Ω the angular momentum. In figure 1.4 the potential is plotted for $M_1/M_2 = 2$.

Conservation of angular momentum dictates the inflow of matter via an accretion disk. In such a disk, matter is transported inwards while its angular momentum is transported outwards at the expense of a small amount of matter. However, some of the angular momentum will reach the compact star and increase the angular momentum of the object. This can go directly, by matter falling on the compact star carrying angular momentum, or indirectly, by torques from tidal effects.

Wind accretion happens when the companion star experiences a mass loss due to a stellar wind. This mass is blown out of the gravitational potential of the companion star, and can then be

trapped in the potential of the compact star. Again, conservation of angular momentum generally leads to spin-up. However, wind accretion is only important for stars with a strong stellar wind. These are generally massive stars or stars in the final stages of evolution. Massive stars have a very short life span leaving little time to spin up a compact object. Stars in the final stage of evolution will expand to much larger radii, often also leading to Roche lobe overflow.

Low mass X-ray binaries

The term low mass indicates a low mass companion of the compact object. In low mass binaries the companion evolves slowly, leaving much time for accretion. As stellar winds are negligible from low mass stars, Roche lobe overflow is the main accretion mechanism in these systems. Compact objects recycled in such a system thought to be the fast millisecond pulsars.

Low mass X-ray binaries are often transient binaries. In transients binaries, accretion occurs semiperiodically in outbursts, separated by periods of quiesence in which little or no accretion occurs. The explanation of this behaviour is that accretion rates are faster than the rate of mass transfer. In such a scenario, accretion starts when an instability develops in the accretion disk. The compact star accretes matter faster than the companion transfers new matter to the disk. The accretion goes on until the disk is completely accreted. During quiesence the disk is then slowly filled until a new instability generates an outburst.

During the transient phase mass transfer will produce large amounts of X-rays, while also heating up the surface of the compact star. During quiesence the compact star will cool down by emitting X-rays. The cooling curve of such objects can be used to study the structure of the crust. This is studied in the thesis of Degenaar (2006), where she concludes that the theoretical understanding of crust cooling seems quite good, but needs further testing against observations. The theoretical results do agree with current day observations but more accurate observations would allow for better testing.

1.1.3 Surface radiation of compact stars

Direct observation of radiation emitted from the compact star surface is currently possible using the X-ray observatories circling Earth. This radiation can be used to study the compact star photosphere. Using the spectrum, much can be learned about the precise structure of the crust and the atmosphere. However, current day understanding of the spectrum is quite poor.

The emission profile also provides information on the cooling of the compact star. These cooling properties can also be calculated from the equation of state and thus provide another mechanism, besides mass-radius determination, to test equations of state. Such a calculation is done in Page et al. (2006).

Neutron star cooling can occur by heat transport of neutrinos and photons. Let us first consider neutrino cooling, as that is the most efficient mechanism in hot neutron stars. After fusing up to iron, the core temperature of a star is usually over 10^{10} K. At these high temperatures, strong neutrino emission will rapidly cool the star. Also when the star is much colder, neutrino emission will still be the dominant cooling mechanism. Only for very low temperatures $\propto 10^6$ K reached in about a million years, photon cooling will become the dominant cooling mechanism. By calculating neutrino cooling, and obtaining the heat capacity and conduction from the equation of state, it is possible to calculate the thermal emission from compact stars as a function of their age (Page et al., 2006).

Since young compact stars often have their supernova remnants still around them, it is possible to find the age of the compact star from the age of these remnants. These ages can be quite accurately determined if the distance to the object is known. From the Doppler shift the velocity of the gas in the supernova remnant is measured. From the distance and the angular size of the remnant an absolute size of this remnant is determined. Using this information and the measured speed the age of the remnant can be calculated.

Sometimes the age of a supernova can also be obtained from historic astronomical data. A famous example is the Crab pulsar, where according to Chinese astronomical archives a supernova has occurred in 1054.

1.1.4 Free precession

Evidence from pulsar timing residuals indicate that long period free precession of the rotation axis of a compact star may occur (Stairs et al., 2000). Free precession occurs as a reaction to a non-spherical disturbance of a rotating object. The precession period is proportional to the oblateness of a compact star. However, it is also proportional to the amount of angular momentum stored in a superfluid coupled to this precession (Shaham, 1977). This makes precession a good probe for the interior properties of a compact star.

The usual model of superconductivity in a neutron star is one in which the superfluid and the magnetic field are closely interlocked. As the crust is made of mostly iron, the crust is coupled

strongly to the magnetic field as well. Superfluids have the peculiar properties under rotation, that lead to strong damping of precession when coupled to the crust via the magnetic field. The occurrence of long term precession indicates that this model may be flawed. Solutions to this problem are a different model of superfluidity, or the occurrence of quark matter and colour superconductivity which has a different coupling to magnetic fields. Superconductivity and colour superconductivity are reviewed in chapter 6. In chapter 7 the consequences of electromagnetic and colour superconductivity on the precession properties of a compact star will be studied.

1.2 Formation and evolution of compact stars

1.2.1 Stellar evolution and the formation of compact stars

Normal stars owe their stability due to the presence of a thermal gradient which balances the gravitational pull. The thermal gradient is maintained by nuclear fusion in the centre of the star. For about ninety percent of its life, stars reside in the main sequence stage. In this stage burning hydrogen in their core is the main source of energy, forming helium in the process. Our sun, for example, is in this stage. A profound property of stars in the main sequence is that they are very stable. Major changes in the characteristics of the star do not happen.

When a star runs out of hydrogen, helium burning has to set in. Since helium has a larger positive charge, the electromagnetic forces between the nuclei will be much larger. In order to make it still possible for the nuclei to come within the range of nuclear forces, a much larger temperature is needed. To reach this temperature in the core, a steep temperature gradient is needed. To compensate this temperature gradient, gravity must provide a compensating pressure gradient. This is possible only for stars more massive than about $0.4 M_{\odot}$.

If a star is lighter than this mass, a degenerate core of helium will form, while the outer layers will be expelled by stellar winds creating a planetary nebula. The core will from a helium white dwarf. Note that such light stars have extremely long lifetimes, longer than the current age of the universe. Single star evolution therefore does not allow for the presence of helium white dwarfs in our current universe. That they do exist is the result of binary evolution, in which mass transfer between the two stars can seriously alter the evolution path of a star (see section 1.1.2).

Stars more massive than 0.4 M_{\odot} can burn helium. Lighter stars, with masses up to about 2 M_{\odot} , burn helium with degenerate cores where the temperature is low compared to density. In such a system, the average kinetic particle energy is much larger than the average thermal energy. Degeneracy is discussed in more detail in chapter 4. In degenerate matter, pressure is not related to temperature. This will lead to runaway nuclear burning: the helium will burn and heat up the star until degeneracy is lifted. This process is called the helium flash. Outside the helium core there is still hydrogen present, leading to hydrogen-shell burning. Heavier stars can stably burn helium in their core. Also these stars will burn hydrogen in a shell around their core. Later, unstable helium shell burning will also occur, accompanied by large mass loss. The end products of helium burning are carbon, nitrogen and oxygen.

Only stars more massive than $M \gtrsim 8M_{\odot}$ can use carbon, nitrogen and oxygen as fuel. These stars have cores more massive than 1.4 M_{\odot} , which can overcome the electron degeneracy pressure and so become dense enough to burn these elements. This mass is called the Chandrasekhar limit and is caused by the failure of the electron degeneracy to effectively counter a strong gravitational pull. The denser the star, the more closely packed the electrons must be. This will eventually cause the Fermi energy, the energy at which the highest energy electrons reside, to go above the electron rest mass. The electrons then become relativistic. Chandrasekhar showed that this leads to a maximum mass for an object prevented from collapsing by electron degeneracy pressure. The outline of his derivation can be found in 2.1.1.

Because the cores of massive stars can become hot enough to burn carbon, nitrogen and oxygen, further fusion will occur. Again, shell burning of lighter elements happens, often off equilibrium. Heavy mass loss is common during this phase. Almost always, stars of this mass are able to burn all elements lighter than iron. However, no star can burn iron to heavier elements. The reason is that iron has the most binding energy per nucleon (figure 1.5). Creating heavier nuclei than iron does not release any energy. A star with a core of iron will therefore have no source of energy anymore.

Since the core is also too massive for electron degeneracy pressure to sustain balance, the core will collapse beyond the white dwarf stage. The core will collapse until a new mechanism is able to provide a balancing pressure. Neutron degeneracy is able to provide this pressure



Figure 1.5: *Binding energy per nucleon. As iron has the maximum binding energy per nucleon, this is the most stable element (image from , NASA)*

against a much stronger gravitational pull (see Shapiro and Teukolsky, 1986). Also compact stars made of free quarks seem to be able to provide the required pressure¹. Free quark matter is a state of matter in which quarks behave as weakly interacting particles. In normal matter, quarks are strongly bound in baryons as a result of confinement, the effect that no quark can isolated. However, at high energies, the asymptotic freedom present in the theory of quark interactions indicates that deconfinement has to occur. Asymptotic freedom means that in the limit of infinite energies the interaction strength goes to zero, as explained in section 3.3. The study of this quark pressure will be the main subject of this thesis. Objects for which neutron or quark degeneracy pressure is the stabilising factor will be very small, with radii of the order of ten kilometres. This is much smaller than typical radii of white dwarf stars, which are or order 10^4 km, in which electron degeneracy pressure counters the gravitational pull.

1.2.2 Supernovae

The collapse of a massive star creates a violent burst, a supernova. When a stellar core collapses, the collapse releases huge amounts of gravitational energy. According to the virial theory, half of this energy will be converted to heat. The massive heating this causes creates a shock wave, which converts all iron in the core of the proto-compact star back to protons and neutrons. How this shock wave is able to have so much energy to be able to do this conversion and also trigger the explosive removal of the outer layers is currently not understood. Probably the neutrino flux caused by weak processes and cooling of the superheated matter plays an important role. It can be calculated that the free path length for neutrino in a very hot proto-neutron star is shorter than the radius of this star (Reddy et al., 1998). This makes it possible for neutrinos to further heat up the proto-neutron star,

¹Publications include Alford et al. (2006a), Andersen and Strickland (2002), Buballa (2005), Fraga et al. (2001), Fraga and Romatschke (2005), Fujihara et al. (2006), Glendenning (2000), Nicotra et al. (2006), Rajagopal and Wilczek (2000), Rüster and Rischke (2004), Weber et al. (2006), Witten (1984)

possibly driving the shock wave.

When more and more neutrons are formed due to beta decay, efficient cooling mechanisms become unavailable as the main neutrino emission mechanism, direct Urca, is unable to conserve simultaneously energy and momentum. This causes the free path length of neutrinos to increase dramatically, making the compact star virtually transparent for neutrinos. As calculated in Page et al. (2006) efficient neutrino cooling via the direct Urca process is only possible when the proton content of a compact star is larger than ~ 11 percent. This is discussed in more detail in section 1.2.4. There also the direct Urca process is described.

It is also possible that the neutron degeneracy pressure or quark degeneracy pressure will not be sufficient to counter gravity. In that case no known mechanism will resist the object from collapsing. Then, the formation of a black hole is expected.

Supernova observations





Supernovae occur about three times per century in an average galaxy (Cappellaro et al., 1999), probably a similar rate occurs in our Galaxy. However, the latest certain supernova observation of a supernova within our Galaxy dates from 1604. Although the explanation lies partially in statistics, also the optical extinction due to dust in our galaxy blocking radiation from supernovae contributes. Furthermore, Cassiopeia A is probably formed after this date but historic observations are not very convincing. An explanation for this is that the luminosity of the supernova forming Cassiopeia A was very low, as has recently been found in some supernovae in other galaxies as well (Tominaga et al., 2005). Yet today, with the huge number of galaxies currently within visible range of our telescopes, several tens of them are discovered each year.

On February 23, 1987 the first and at the time of writing this thesis only nearby supernova occurred that could be studied with modern instruments. At that day, a blue giant star exploded in the Large

Chapter 1: Introduction

Magellanic Cloud, at a distance almost 170,000 light years. This supernova was visible to the naked eye, and it was also the first object outside our solar system from which neutrinos have been identified as belonging to this object. About three hours before the visible supernova explosion occurred, a 13 second neutrino burst was observed (see figure 1.7). This result indicates that it takes about three hours for the shock wave to propagate to the surface of the star, matching model calculations. Furthermore, this event allowed for accurate testing of some neutrino properties. In the remnant of supernova 1987A, as this event is called, no compact star has yet been discovered. It is expected that in due time the compact star will be found, as these neutrino observations are in agreement with cooling models of a proto-compact star.

In other supernova remnants compact stars have been discovered. Both the Crab nebula (see figure 1.1) and Cassiopeia A host a compact star. The Crab nebula is associated with a supernova that went off in 1054 and was observed by Chinese astronomers. Cassiopeia A probably went supernova around 1680, although no clear recording was made in that year. There is a record of the appearance of a new star in 1667, which may be the supernova related to Cassiopeia A.



Figure 1.7: *The detection of neutrino's by the Japanese Kamiokande II and the IMB experiment (image from UNO collaboration, 2001)*

1.2.3 Magnetic fields

One of the most important characteristics of a compact star is its magnetic field. Ranging from 10^4 T for millisecond pulsars up to 10^{12} T for magnetars, these strong magnetic fields have an important influence on the compact star. They are the main cause of pulsar emission, as discussed in section 1.1.1. In binary systems the magnetic field of the compact star will interact with the accretion disk. This can lead to very complex behaviour of accretion and may be related to some of the quasi periodic signals the objects emit (Boutloukos et al., 2006). It also tunnels the accretion to the magnetic poles, generating hotspots on these poles. The X-ray production occurs in these hotspots leading to pulsed X-ray emission with the frequency of the neutron star its spin.

Magnetic fields are thought to be present as a result of magnetic flux conservation. Massive stars, capable of forming a compact star, have magnetic fields present. These fields are normal stellar magnetic fields, thus much weaker than compact star fields. During the collapse the magnetic flux is preserved. As a result of the order 10^6 decrease of the radius the magnetic field is boosted by an order of 10^{12} . Assuming a field similar to our sun, this would lead to a 10^9 T field. These fields perfectly match the fields of normal compact stars.

Recently it has been discovered that there probably is a significant difference in magnetic fields of the stars that form compact stars, the B stars² (Ferrario and Wickramasinghe, 2005). There exist high field B stars, having fields of an order 100 stronger than normal B stars. It is thought that these stars might evolve to magnetars when they collapse into a compact star. This story is similar to strong field white dwarfs, which make up a similar fraction of the total number of white dwarfs as the fraction of strong field white dwarf forming A stars to normal A stars, the stars that create these white dwarfs. This suggests that the strong field stars evolve into strong field white dwarfs.

²The letter represents the spectral type of the star, which is related to its surface temperature. B stars have a surface temperature $\propto 2 \times 10^4$ K, A stars $\propto 1 \times 10^4$ K. The complete list from hot to cold is O B A F G K M, the sun is of type G.

An observational challenge is provided by the low number of massive stars. Massive stars are much more seldomly produced, as found by Salpeter based on a statistical analysis of the local stellar population (Salpeter, 1955). The fitted power law relation is

$$M_0 \propto M^{-3.5} \tag{1.3}$$

Also, they are very short lived, with a life span $\propto 10$ million years. As a result massive stars are not a frequent occurrence, making statistics much more difficult. Another problem is the so called selection effect, that causes high field compact stars to be found much more easily. Due to these problems the situation for compact stars is not very clear cut.

Another possibility for creating the strong field compact stars, is by invoking a dynamo action during its collapse. Although not known how this should work, it is not that strange an idea that the massive friction during a collapse could in principle create huge magnetic fields. A drawback from this view is that it is not clear why in some supernovae this dynamo action would be present, while in others it has to be absent to predict the normal compact stars.

For the millisecond pulsars another mechanism has to be invoked. These compact stars are characterised by a, for compact stars, weak field of about 10^4 T. The magnetic breaking as a consequence of its field is such that the time scale for a significant change in the rotation period is larger than the Hubble time. Therefore, it can be assumed that the periods at which these stars rotate haven't significantly changed during their lifetimes.

As discussed earlier, millisecond pulsars are probably generated by accretion in binary systems. How accretion leads to weak fields is still subject of much debate. One possibility is that the compact star first spins down to a very slow period. However, this result is contradicted by observations of slowly rotating compact stars in binary systems which still posses normal magnetic fields $\sim 10^8$ T (Coburn et al., 2006). Another possibility is that the magnetic field is buried by accretion. It is then still present, but it doesn't emerge from the surface and is therefore not observable (Bhattacharya and Srinivasan, 1995). None of the models is currently able to provide an established mechanism to explain the weak fields of millisecond pulsars, which seems to cluster at around 10^4 T (see figure 1.3).

Evolution of magnetic fields in isolated pulsars

As can be seen in figure 1.3 the lifetime of a pulsar strongly depends on the strength of the magnetic field. High-field pulsars are expected to produce pulsar-radiation for only about ten million years, while the low field millisecond pulsars can in principle have existed since the formation of the universe, as their life span based on spindown timescales is longer than the Hubble time.

After the discovery of a few dozen pulsar systems, there seemed to be evidence for the decay of the magnetic field of the high-field pulsars with a decay time similar to its lifetime, $\propto 10$ million years. After the discovery of many more pulsars the evidence for the decay of the magnetic field has diminished, leading to the opinion that these fields are stable for much longer than the pulsar lifetime (Bhattacharya and Srinivasan, 1995). The pulsar lifetime is defined as the time pulsars are capable of producing electromagnetic radiation by the pulsar mechanism. As seen in figure 1.3 it seems that if pulsars rotate too slow this mechanism no longer works. For pulsars with a normal field $\sim 10^8$ T this occurs for rotation periods of about 10 seconds.

As discussed in section 1.1.1 one of the possibilities for the stability of the magnetic field is the presence of a superconducting medium that is coupled to the magnetic field. Such a system would only show a very slow decay of the field, well within observational bounds. A detailed analysis is provided in section 7.1. Yet, the superconducting picture is not necessary, as degenerate matter is a very good conductor anyway. For neutron matter this conductivity is about 10^{28} s⁻¹, leading to decay

Chapter 1: Introduction

times ~ 10^{12} yr, larger than the Hubble time (Baym et al., 1969). For quark matter it can be calculated that Ohmic decay times are of similar order ~ 10^{13} yr (Alford et al., 2000). Ohmic decay times in the crust are much shorter (Cumming et al., 2004). See Harding and Lai (2006) for a recent discussion.

Field decay in binary systems

In contrary to solitary systems, fields of compact stars in binary systems do decay. Evidence for this decay is provided by observations of X-ray binaries. In high-mass X-ray binaries, the compact star is often pulsating, suggesting a strong field able to generate pulsed X-ray emission. In contrast, the compact star in low-mass X-ray binaries is seldom a pulsating object, an indication for much weaker fields. Since the high-mass binaries are generally much younger than the low-mass binaries, this can be seen as an indication of field decay. This indication is strengthened further by the low fields of millisecond pulsars, for which there is good evidence that they originate from the low-mass X-ray binaries (Bhattacharya and Srinivasan, 1995).

However, as these object undergo mass transfer there might be other mechanisms at work altering their magnetic field. Mass transfer severely heats up the crust, and also alters its equation of state. This might lead to an extra Ohmic decay component. Yet, it is unclear how such a mechanism is capable of reducing the core magnetic field. Suggestions are that these fields are somehow buried, although the mechanisms involved are still very unclear.

Magnetic field evolution of magnetars

Due to the extremely high magnetic fields of magnetars $(B \sim 10^{10} - 10^{12} \text{ T})$ the evolution of their magnetic fields is different. Part of the radiation emitted by these objects is probably generated by the decay of the field, as the total energy emitted is larger than the observed spin-down.

If the matter inside the object is of hadronic origin, the field probably exceeds the critical field for superconductivity. This might present a mechanism for Ohmic dissipation. Also reconnecting field lines can liberate much energy. Such events are thought to be the origin of giant outbursts as are observed in the soft gamma ray repeaters (SGR). How Ohmic decay can be strong enough to explain the decay is not clear.

If the interior is made up of quark matter the critical field of colour superconductivity will not be reached (see section 6.2, Alford et al. 2000), so there is no comparable mechanism available for field decay in compact stars made of quark matter.

1.2.4 Cooling of compact stars

The age of a compact star is important, as knowledge of the age allows for testing evolutionary models of the compact star. When a compact star is born, it is initially very hot, with temperatures larger than 10^{10} K. This large temperature is the result both from the supernova as well as the initial core temperature, which must be of order 10^{10} K to allow burning up to iron in the first place. At these temperatures, nuclei become unstable and can be thermally destroyed. This results in the breakup of all nuclei to protons and neutrons. The degeneracy pressure of the electrons then makes it energetically favourable to convert electrons and protons into neutrons via inverse beta decay

$$p + e^- \to n + \nu_e \tag{1.4}$$

resulting in the production of a neutrino flux. The reverse of this process, beta decay, is also possible: $n \rightarrow p + e^- + \overline{\nu_e}$. The two processes combined are able to convert heat into neutrinos at a very fast pace. This is called the Urca process, after a Brazilian casino where legend says that it is possible to loose money at an equally fast rate.



Figure 1.8: Production of neutrino bremsstrahlung in a quark scattering diagram

This neutrino flux quickly cools the neutron star to temperatures $\propto 10^9$ K, when the inverse beta decay process becomes unavailable due to the degeneracy of the neutrons blocking momenta of the final state fermions. In order for the direct Urca process to occur the momenta of participating particles must lie close to the Fermi surface. Lattimer et al. (1991) have shown that this condition is met only when the proton fraction exceeds 11%.

Below this threshold stage, modified Urca processes become the dominant cooling mechanisms. This process requires an additional spectator baryon to carry away the momentum. As this is a three particle process, the rate will be much slower. Other processes cooling a neutron

star at this stage include neutrino-bremsstrahlung (Price, 1980). In this process, a baryon is accelerated via the nuclear interaction and a neutral current Z_0 is produced. The Z_0 then decays very quickly to two neutrinos. This process is very similar to the electromagnetic bremsstrahlung in particle accelerators, except of course that the so produced photons are a stable final state.

It is also possible that in the compact star free quark matter will be present. This leads to a somewhat different cooling behaviour. However, since quarks also carry isospin charge, the main characteristics are the same. Quarks also undergo direct or modified Urca cooling, and also quarks produce neutrino bremsstrahlung. The difference is that in a state in which free quark matter is present there are more particles. More particles generally leads to faster cooling, and thus the expectation that quark stars are somewhat cooler than stars consisting only of neutrons.

The story above is complicated by the presence of superfluid states. Both quarks and neutrons will probably undergo a superfluid transition at temperatures $\sim 10^9$ K for neutrons and $\sim 10^{10}$ K for quarks. As superfluidity causes a gap between the superfluid ground state and excitations, final states for scattering become even less available. The critical temperature for quark matter is larger than the critical temperature for neutron matter. This may actually cause quark stars to be hotter than their neutron counterparts. Note that the critical temperature is of the order of the formation temperature of quark stars. These objects might thus be in a superconducting state immediately after formation.

Colour superconductivity as present in weakly interacting quark matter can have important influence on the quark equation of state. In chapter 6 superconductivity and colour superconductivity will be explored further. Cooling models of compact stars are studied in Page et al. (2006).

CHAPTER 2

Structure of compact stars

In this chapter the general structure of a compact star is reviewed. First we will derive the general equations of stellar structure. These equations must be satisfied by any model describing a spherical star. In order to solve the models, we will need an equation of state, relating pressure and density. Using a power law equation of state, a so-called polytropic relation, the general properties of stellar models can be studied. We will derive why a polytropic relation is useful and study its behaviour.

In the second part of this chapter we will look at some complications to the picture sketched in the first part. More general equations of state will be considered, and the effect of rotation on the structure equations is studied.

2.1 General structure equations

The structure of a compact object is determined by its equation of state and the equations for hydrostatic equilibrium. In this thesis, we will deal with massive objects with small radii, so taking the weak field limit for the gravitational potential is not appropriate. Therefore, we cannot use Newtonian formulae, but have to use a general relativistic formulation.

For a spherical, non-rotating star in equilibrium the relativistic hydrostatic equation was first derived by Tolman (1939) and Oppenheimer and Volkoff (1939). It can be derived from the Schwarzschild metric using a nonzero energy momentum tensor (see appendix A for the derivation). The TOV equation reads

$$\frac{dP}{dr} = \frac{G(\epsilon + P)(m(r)c^2 + 4\pi r^3 P)}{r^2 c^4 (1 - 2Gm(r)/(c^2 r))}$$
(2.1)

In this equation c is kept explicitly, in order to understand the relation with Newtonian mechanics. The pressure P and the energy density ϵ are both given in units of energy per volume. The enclosed mass m(r) is given by equation

$$\frac{dm(r)}{dr} = 4\pi\epsilon \frac{r^2}{c^2} \tag{2.2}$$

with again ϵ the energy density. Note that as $c \to \infty$, the TOV equation asymptotically approaches the equation of hydrostatic equilibrium based on a Newtonian gravitational potential, as it should. Then,

in equation 2.2 the energy density ϵ has to be replaced by the matter density ρ .

$$\frac{dP}{dr} = \frac{Gm(r)\rho}{r^2} \tag{2.3}$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho \tag{2.4}$$

Note that M(r) defined by

$$M(R) = \int_0^R \frac{\mathrm{d}m}{\mathrm{d}r} \mathrm{d}r \tag{2.5}$$

is the total gravitational mass. Generally, the gravitational mass of a compact star is reduced with respect to the integrated energy density due to its gravitational binding energy. The difference between gravitational mass and integrated energy density can be quite significant. This difference is taken into account by the metric part of the Einstein equation. More on this can be found in appendix A.

The equations of hydrostatic equilibrium must be supplemented by an equation of state to form a closed set of equations. The equation of state is non-trivial and requires knowledge of the composition of the object. In case of a perfect fermion gas an analytic solution to the equation of state is possible. In the limit of non-relativistic or extreme relativistic degenerate gases the equation of state reduces to a power law dependence of pressure P and density ρ . This is know as a polytropic equation

$$P = K\rho^{\gamma} \tag{2.6}$$

with K a constant and γ the power of the polytrope. For non-relativistic polytropes $\gamma = 5/3$, while $\gamma = 4/3$ for relativistic polytropes.

2.1.1 Polytropes

The polytropic law can is obtained using an ideal degenerate gas. Such a gas has an energy density

$$\epsilon(p) = \sqrt{m^2 + p_F^2} \tag{2.7}$$

where p_F is defined as the Fermi momentum. Taking $x \equiv p_F/m$ in this equation, we get

$$\epsilon(z_F) = m[\sqrt{1 + z_F^2} - 1]$$
 (2.8)

These equations can be used to write down the number density

$$n = g(p_F)m^{-3} \int_0^{z_F} 4\pi z^2 dz = \frac{4\pi g(z)}{3m^3} z^3$$
(2.9)

Here $g(p_F)$ is the number of states per Fermi level. For spin-1/2 particles $g(p_F) = 2$. Similarly, the pressure is given by

$$P = g(p_F)m^4 \int_0^{z_F} \frac{z^4 dz}{(1+z^2)^{1/2}}$$
(2.10)

This equation is conveniently written as Af(z), where A is a dimensionful constant while the dimensionless function f(z) is

$$f(z) = z(2z^2 - 3)(1 + z^2)^{1/2} + 3\sinh^{-1}(z)$$
(2.11)

In the limit of small or large z we obtain from this equation the limits:

$$P \propto \left(\frac{\epsilon}{\mu}\right)^{5/3} \quad z \ll 1$$
 (2.12)

$$P \propto \left(\frac{\epsilon}{\mu}\right)^{4/3} \quad z \gg 1$$
 (2.13)

A careful derivation of the equation of state for an ideal degenerate gas is provided in section 4.1.

Together with the equations for hydrostatic equilibrium and mass conservation this equation can be solved to obtain the mass and radius of a star as a function of central pressure. For non-relativistic degenerate gases, the radius will decrease with increasing mass, while for the relativistic case the dependence on central pressure drops out and a maximum mass appears, as will be shown below.

When using the Newtonian equation of hydrostatic equilibrium and a polytropic equation of state, there exists an exact solution. Combining equations 2.3 and 2.2 yields

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho \tag{2.14}$$

while the polytropic equation of state (equation 2.6) can be written in dimensionless form

$$\rho = \rho_c \theta^n \tag{2.15}$$

$$r = a\xi \tag{2.16}$$

$$a = \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}$$
(2.17)

with $\rho_c = \rho(r = 0)$ and *n* defined as

$$\gamma = 1 + \frac{1}{n} \tag{2.18}$$

Using the dimensionless equations 2.15-2.17 to rewrite equation 2.14 one obtains

$$\frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\theta^n \tag{2.19}$$

This equation is called the Lane-Emden equation. The boundary conditions that close the set of equations can be obtained by requiring that $\rho(0) = \rho_c$ and $\rho(r) = 0$. These conditions are $\theta(0) = 1$ and $\theta(1) = 0$.

Its use to dense structures is that it gives an upper bound to stellar masses and radii, as general relativity acts as an extra attractive interaction at high densities. The functions for mass and radius obtained from the Lane-Emden equation are

$$R = a\xi_1 \tag{2.20}$$

$$M = 4\pi a^3 \rho_c \xi_1^2 |\theta'(\xi_1)|$$
 (2.21)

which combine to

$$M = 4\pi R^{(3-n)/(1-n)} \left[\frac{(n+1)K}{4\pi G} \right]^{n/(n-1)} \xi_1^{(3-n)/(1-n)} \xi_1^2 |\theta'(\xi_1)|$$
(2.22)

where ξ_1 is defined by $\theta(\xi_1) = 0$, which can be solved numerically from equation 2.14.

For a relativistic polytrope ($\gamma = 4/3$) the solution for *M* is independent of the central density. Using the numerical results $\xi_1 = 6.89685$ and $\xi_1^2 |\theta'(\xi_1)|$ (Shapiro and Teukolsky, 1986)

$$M = 1.457 \left(\frac{2}{\mu_e}\right)^2 M_{\odot}$$
 (2.23)

with μ_e is the mean molecular weight. This limit can also be obtained by an energy balance between gravitational binding energy E_{grav} and the internal energy of the relativistic Fermi gas E_{int} (Shapiro and Teukolsky, 1986)

$$E = E_{\rm int} + E_{\rm grav} = \frac{N^{1/3}}{R} - \frac{GNm_b^2}{R} = 0$$
(2.24)

which yields a similar mass as equation 2.23.

The advantage of this equation is that it is easy to add corrections to internal energy and the gravitational potential. In this manner, it is possible to correct equation 2.24 in such a way that general relativity is accounted for. The correction factor is calculated from the TOV equation in Shapiro and Teukolsky (1986) and is

$$\Delta E_{GR} = -0.918294 M^{7/3} \rho_c^{2/3} \tag{2.25}$$

which enhances gravitational binding, as expected. Note that this factor is no longer correct when other corrections to equation 2.24 are made. It will, however, still be true that general relativity deepens the gravitational potential well.

Taking into account other interactions than gravity is also possible. This is important, as an ideal gas assumption works quite well to describe white dwarfs, but models based on an ideal gas for neutron and quark stars do not agree with observations. This can be understood by realising that interactions between neutrons and quarks cannot be considered small, so an ideal gas approach is not adequate. A simple solution is to express the mean field interactions as a vacuum pressure and include this term. This will be discussed in section 5.1.1.

2.2 Modelling stars

2.2.1 Models of neutron stars and quark stars

There are typically two distinct models of neutron and quark stars, and a number of adaptations to this. A schematic overview of these models is given in figure 2.1. The oldest model is the pure neutron matter model. This model consists of a crystalline iron crust, which becomes more neutron rich deeper in the star. At a certain depth the neutrons will drip out, forming a neutron fluid. This fluid is probably a superfluid, a fluid that can flow without friction but also has some peculiar properties when rotated. When rotating a superfluid, not the superfluid itself will rotate, but all angular momentum will be contained in small vortices where the matter is not in a superfluid state. This is a result of the broken rotational symmetry due to long-range order in the superfluid state.

The vortices will pin to impurities as that results in a state of lowest energy. In a neutron star, these impurities are provided by the crust lattice. Movement of pinned vortices can only occur by vortex creep, a process in which vortices tunnel through the energy barriers created by pinning (Anderson, 1962). Pinning to the lattice and an inability to have a vortex creep in a certain region of the star can be used to explain glitches (section 1.1.1). Vortex pinning and its consequences for glitches will be further explored in section 7.2.1.

Deeper in the neutron star atomic nuclei become unstable, so also free protons appear. These protons are thought to form a superconducting fluid. The appearance of superconductivity has important influences on the behaviour of the magnetic field of the neutron star, which will be the subject of chapter 7.

In the neutron star centre, very high densities may lead to deconfinement. Then, the dense interior must be described using a quark equation of state. We will call such a star a hybrid star. It is also possible that other forms of matter may be present in the interior, like strange quark containing baryons and mesons (Hyperons and Kaons) as well as drops of strange matter.

Another interesting possibility is that when strange matter is created the energy released by the conversion may actually trigger the whole star to convert to strange matter (Witten, 1984). This object is called a quark star or strange star. These objects may be visible as bare quark stars, but it is more likely that they will have a very thin crust. Such a crust would appear if charge neutrality is not present in the quark matter. This is likely due to the mass difference of the up, down and strange quarks. In this case, electrons must be present for charge neutrality. At the surface, the quark density will drop to zero over a very short distance of a few femtometer. This is much faster than the electric field falls off, leading to a strong electric field. Such a field would be able to support a crust with a mass $\propto 10^{-5} M_{\odot}$ (Alford et al., 2006b, Jaikumar et al., 2006b, Stejner and Madsen, 2005). In this case the quark star will probably have an emission very similar to that of a neutron star.

2.2.2 Solving the models

The equations of state and the structure equations 2.1 and 2.2 yield, at zero temperature, a closed set of equations given two boundary conditions. For nonzero temperatures equations to describe heat loss and energy production in the star are also needed. The dependence on temperature is of minor importance for both neutron stars and all conceivable quark star models, since the interaction energy is generally much higher than the thermal energy available in such stars. It is therefore safe to assume zero temperature when determining the structure of compact stars.

Analytically solving this set of equations is in general not possible. For a polytropic equation of state there do exist exact solutions (see for example Shapiro and Teukolsky, 1986), but for a more realistic one numerical solutions have to be used. Solving the set of equations is done by first assuming an arbitrary central pressure as the first boundary condition. Using the structure equations one can then find a new pressure a distance δx from the origin. Repeating this process until the surface is reached yields a model of the star. The location of the surface is determined by a boundary condition, generally the surface is assumed to exist at the radius where the pressure vanishes. A detailed analysis is provided in chapter 5.

In the structure equations 2.1 and 2.2 a non-rotating, spherical star is assumed. Of course a more general set of equations can be used, making the problem to be solved more complex and thus more expensive in computing time. However, if these effects are of great influence, they should not be neglected. The effects of rotation are studied in section 2.2.4, and are found to be of some importance for the millisecond pulsars. However, the effects of rotation will only lead to small corrections and can be neglected as the errors resulting from uncertainties in the equation of state are much larger.

Due to the extreme gravitational pull of a compact object, the structure will be completely determined by the shape of the potential well. In case of a non-rotating star, a spherical potential is the state of minimum energy. Since it can be assumed that surfaces of equal pressure match equipotential surfaces in such strong potentials, the star needs to have a spherical shape. The assumption that the potential determines the shape means that gravity is much stronger than long-ranging structural forces that can exist in materials. As this is already the case for planets, it is certainly the case for compact objects, so a spherical potential is a good assumption.

2.2.3 Equations of state



Figure 2.1: Possible model of a compact star. The radii of the different components may vary, depending on the model used and the object mass. Not shown is the structure that might be present in between the crust and proton-neutron fluid. Here a complicated area might be present, were fore example vortices of the neutron superfluid can interact with the crystal lattice structure of the iron crust.

In section 2.1 the equation of state of a degenerate ideal gas was studied. We found that in the limiting case of relativistic or non-relativistic equations, simple polytropic behaviour occurs. In intermediate cases, a linear combination of these limiting cases will apply.

However, matter found in compact objects is not an ideal gas. While the Coulomb forces in white dwarf stars may be weak and therefore considered to be a minor perturbation to the ideal gas approximation, in denser neutron stars and quark stars the interparticle forces are much stronger. This leads to a serious complication of the problem of finding an equation of state. Another problem is that the behaviour of dense matter is poorly known, as it is impossible to study such matter on Earth. In experiments at GSI in Darmstadt and RHIC in Brookhaven heavy ions are collided in order to obtain high densities, but these experiments are unable to obtain supernuclear densities at low temperatures (Buballa, 2005). The ALICE experiment at the LHC in Geneva will suffer from the same limitations. The result is that there is no equation of state for neutron matter that is generally accepted.

The strongest force in neutron stars and quark stars is the strong force. A major characteristic of the strong force is asymptotic freedom as described in section 3.3. It might be that due to the high densities inside neutron stars the strong coupling constant becomes so small that deconfinement occurs. Although deconfinement cannot be observed at high densities and low temperatures, at high temperatures it can be observed in particle accelerators. There is evidence that in RHIC the quark-gluon plasma, as the high temperature state is called, has been observed (Gyulassy and McLerran, 2005), although the result is still controversial (Star Collaboration, 2005). Also in the early universe this quark-gluon plasma must have been present.

While the strong force will be studied in chapter 3, it is interesting to consider the qualitative

consequences of this possibility. It is important to realise that quarks are light particles. The Fermi energy of quarks in a baryon is about 200 MeV, which is already much larger than the likely mass of up- and down quarks (both around 5 MeV). This energy is estimated using a baryon size of 1 fm. The uncertainty principle requires momentum and energy to be at least \hbar . Using that $\hbar c \approx 200 \text{ MeV}$ fm, the uncertainty principle indicates that a particle in a 1 fm box needs an energy of at least 200 MeV. Since this energy is much larger than the rest energy, it is safe to consider these particles massless in equations. The strange quark is more massive (about 100 MeV), but with typical Fermi momenta achieved at high densities (at least 200 MeV) this particle will also become relativistic. However, ignoring the strange quark mass does not seem justified as will be shown in chapter 5

For a model of relativistic particles to provide stars with a range of masses, it is vitally important to include their interactions. Otherwise the result will be a relativistic ideal gas, which allows one solution only, as shown in 2.1. The equation of state of weakly interacting quark matter leads to a completely different behaviour of the mass-radius relation. At lower masses, quark stars will grow if the star becomes more massive, indicating that quark matter behaves as an incompressible medium. Only near the maximum mass a quark star can have an increase in mass leads to a smaller radius, just as neutron stars. The details of the quark star equation of state will be studied in chapter 4 and 5.

2.2.4 Effect of rotation on compact star structure

Some compact stars are known to spin very rapidly (see section 1.1.1). In this case, it is important to realise that centrifugal forces might be an important influence to the structure. Obtaining an equation for the potential can be achieved by solving the Kerr metric with an ideal fluid energy-momentum tensor. However, this 'Kerr-TOV' equation has currently not been found, leaving only numerical methods to calculate the potential of a spinning compact star.

In Colpi and Miller (1992) it is shown that fast rotation can have profound effect on the massradius relation, leading to larger radii (up to 0.3 larger) for sources rotating at near breakup speeds compared to non-rotating objects. Furthermore, fast periods allow a somewhat higher maximum mass (about 20%). The calculations in Colpi and Miller (1992) were done assuming a star rotating at near breakup speeds. In this section we will estimate the effect of rotation on more slowly spinning compact stars found in nature. We will see that spin can generally considered to be slow, even for the fastest spinning pulsars.

An estimate of the importance of spin can be made by a comparison of the centrifugal force F_c to the gravitational force F_G , with the forces given by (Colpi and Miller, 1992)

$$F_c = r\Omega^2 \frac{1 - 3M_0/r}{1 - 2M_0/r - r^2\Omega^2}$$
(2.26)

$$F_G = M_0 / R^2 \tag{2.27}$$

 F_c is the general relativistic equation for a particle to move around an object of mass M_0 and distance r from the centre. In this equation G = c = 1, and the gravitational force is expressed by a Newtonian equation. Although the weak-field limit is not appropriate for compact stars, the effect of spin will prove to be so small that a careful analysis is not necessary.

Calculating F_c/F_G as a function of Ω and setting it equal to 1, gives $\Omega \approx 2000 \text{ s}^{-1}$, for a 1.4 M_{\odot} compact star with a radius R = 11 km. So a pulsar has to spin at 2000 Hz to reach breakup speed. For a long time the fastest known pulsar rotated at a frequency of 642 Hz (1.6 ms). Recently faster rotating pulsars have been found. In March 2006 a pulsar rotating at 716 Hz (1.4 ms) was claimed (Hessels et al., 2006), which is currently the fastest spinning pulsar known. For a period of 1.4 ms the fraction $F_c/F_G \approx 0.08$. As this fraction is smaller for slower rotating pulsars, the assumption that

a slow rotation approximation can be used seems justified. Yet, these results only apply for objects rotating at near breakup velocities.

In addition to this, the uncertainties in the equation of state of neutron stars and quark stars are larger than a possible alteration due to rotation. Again, thus suggests it is safe to ignore the consequences of rotation. Therefore, throughout this thesis rotation is ignored.

CHAPTER 3

Quantum chromodynamics at high densities

Quarks are charged with respect to all forces. However, the theory of quantum chromodynamics has by far the strongest coupling constant, thereby being of greatest influence on the equation of state of the quarks. Weak nuclear interactions are essential to describe beta equilibrium and cooling mechanisms, electromagnetism is assumed to be of only minor importance. In this chapter we will review some of the quantum field theory needed to understand quarks at high densities.

3.1 Symmetry groups

Symmetries of systems are almost always important. Neuther's theorem states that symmetries are coupled to conserved quantities. Examples are the translational symmetry giving rise to momentum conservation and gauge symmetry leading to charge conservation.

In order to understand the properties of quark matter, it will be useful to use the symmetries of the system. The symmetries discussed are mostly abstract, mathematical symmetries. There will be one exception, however, and that is the symmetry of rotation. Rotational symmetry is broken by a superfluid state (see chapter 7) which may also occur in quark matter.

Rotational symmetry is a fundamental property of unordered isotropic systems. It is best described by stating that some function describing the system should be unaltered by rotations. A useful way is to consider the set of all rotations as a group *G*. This set can be described as follows: a rotation is a transformation of a system that preserves the inner product of two vectors. The set of all operators satisfying this condition in a three dimensional space is called SO(3), the group of special orthogonal 3×3 matrices. Special states that det(*M*) = $1 \forall M \in G$, orthogonal means that $M^{-1} = M^T$.

It is possible to reproduce all the necessary matrices from just three generators. These generators are in essence infinitesimal rotations along the *x*, *y* and *z* axis. The procedure to obtain a group element is to take the exponent of the generators $M = \exp[i\partial \mathbf{\hat{n}} \cdot \mathbf{L}]$ with $\mathbf{\hat{n}}$ the rotation axis and \mathbf{L} the generators of SO(3). In the adjoint representation the generators are

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\phi_{x,y,z}$ are the angles of rotation along the *x*, *y* and *z* axis, respectively.

In a similar fashion we can now also introduce new symmetries. We would like to have a unitary theory so that preservation laws are obeyed, thus we better use unitary operators. Unitary matrices have the property that the hermitian conjugate is equal to the inverse $M^{-1} = M^{\dagger}$. The simplest example of such a group is U(1) defined as

$$M = e^{i\phi} \tag{3.1}$$

Expectation values are globally symmetric under this group, as

$$\langle \psi | \psi \rangle \to \langle \psi e^{i\phi} | \psi e^{i\phi} \rangle = \langle \psi | e^{-i\phi} e^{i\phi} | \psi \rangle = \langle \psi | \psi \rangle \tag{3.2}$$

We will see later in this chapter that, if we require the symmetry of equation 3.1 to become a local symmetry, which means that ϕ depends on x, we need to alter our theory to include electromagnetic interactions.

Of more general unitary symmetry groups we only need to consider det(M) = 1, as all unitary matrices with $det(M) \neq 1$ are just a product of a special unitary matrix with an element of U(1). We choose to consider U(1) separately. The simplest nontrivial example then is SU(2). It is defined as all unitary matrices of the form

$$M = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$$

with ad - bc = 1. Note that in general two elements of this group do not commute: it is a non-Abelian group. It is generated by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It can be shown that SU(2) is locally isomorphic to SO(3), the group of rotations. This explains why angular momentum and spin can be added in atomic physics as if it were the same property. However, the SU(2) symmetry can also be seen as the symmetry of the isospin interaction.

The next relevant group is SU(3), the symmetry group of the strong interactions. It is generated by the eight Gell-Mann matrices, therefore this symmetry will need eight gauge bosons: the eight gluons. Again this is a non-Abelian group. This will lead to charged gauge bosons, causing asymptotic freedom as discussed in 3.3.

3.2 Quantum field theory

Combining quantum mechanics with special relativity requires the concept of a field. In special relativity, observers do not need to agree on observables such as energy. Since this theory also makes it possible to convert energy to particles via $E = mc^2$, a fixed-number particle theory does not work. Needed is a theory describing in principle infinitely many particles, and the most natural choice is then a field theory. In quantum field theory wave functions describing particles become operators, creating particles from the vacuum. In a free theory this leads to amazingly simple equations, such as the Klein-Gordon equation for a free scalar field $\phi(x)$

$$(\partial_{\mu}\partial^{\mu} + m^2)\phi(x) = 0 \tag{3.3}$$

or the Dirac equation for a free fermion field $\psi(x)$

$$(i\partial - m)\psi(x) = 0 \tag{3.4}$$

As usual, the summation over multiply occurring indices is implied. When quantised, these equations result in creation and annihilation operators, identified as the operators creating and annihilating the particles. The negative energies occurring in the equation can be interpreted as antiparticles.

These equations possess global symmetries, like the U(1) discussed in section 3.1. Requiring these symmetries to be also locally valid, it is possible to acquire an interacting theory. For example the U(1) symmetry can be used to derive electromagnetism. Explicitly for an arbitrary symmetry group

$$\phi(x) \rightarrow e^{igM(x)}\phi(x) \tag{3.5}$$

$$\phi^*(x) \to e^{-igM(x)}\phi^*(x) \tag{3.6}$$

$$\partial_{\mu}\phi(x) \rightarrow e^{igM(x)}\partial_{\mu}\phi(x) + i\partial_{\mu}M(x)e^{iM(x)}\phi(x)$$
 (3.7)

were M(x) is an element from a gauge group, and g is the coupling strength. The term containing the derivative breaks gauge invariance. This can be solved by introducing a covariant derivative which does transform in the desired way

$$D_{\mu}\phi(x) \to e^{igM(x)}D_{\mu}\phi(x)$$
 (3.8)

This can be done by requiring

$$D_{\mu} \equiv \partial_{\mu} + igA_{\mu} \tag{3.9}$$

with transformations of A_{μ} satisfying

$$A_{\mu} \to A_{\mu} - \partial_{\mu} M \tag{3.10}$$

This last requirement is exactly the gauge invariance present in massless vector fields, like photon and gluon fields. Introducing this derivative in the free Lagrangian density \mathcal{L}

$$\mathcal{L} = \overline{\psi}(i\partial - M)\psi \to \mathcal{L} = \overline{\psi}(i\partial - M)\psi + g\overline{\psi}\gamma^{\mu}\psi A_{\mu} + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$
(3.11)

leads to an interaction Lagrangian via the interaction term $g\bar{\psi}\gamma^{\mu}\psi A_{\mu}$. Finally, to describe the gauge field behaviour, inclusion of

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ig[A_{\mu}, A_{\nu}]$$
(3.12)

is necessary.

For *M* an element of U(1), the counterterms A^{μ} can be identified as gauge bosons of the electromagnetic field, and also the interaction between particle and photon is present. The commutator in 3.12 vanishes for an Abelian symmetry group like U(1).

In a similar way, SU(2) leads to the weak interactions and SU(3) to the strong interactions. Complication to the latter two theories is that they are formed by non-Abelian symmetry groups (not all elements of the group *G* commute: $\exists A, B \in G : AB \neq BA$). In this case the commutator in 3.12 cannot be removed from the Lagrangian, leading to interactions between the gauge bosons.

For realistic quantum field theories, no analytic solutions are available¹. However, it is possible to write the solution as a series of Green's functions, yielding a perturbation series expansion. Each next order in this series expansion will be of a higher power in the coupling constant. With a small coupling constant higher order terms will soon provide only minor corrections.

The most convenient way to find this series expansion is by using Feynman diagrams. Feynman showed that with only a small number of rules, the Feynman rules, it is possible to write down all

¹Only a few interacting field theories can be solved analytically, and only in two dimensions (see Peskin and Schroeder, 1995, ch. 22, and references therein)

occurring terms. Connecting the external lines in all topological different ways using a maximum of n vertices, n being the order of your expansion, you find all possible amplitudes from the in-state to the out-state at that order. By this method an in principle difficult mathematical problem has been reduced to finding all different diagrams and calculating the absolute square of the sum of diagrams. This is still not an easy task, but at least calculations are possible. Yet, it only provides a perturbative approach. Another drawback is that this approach only works for a coupling constant $\ll 1$.



Figure 3.1: The standard model of particle physics. The above two rows are the quarks. These particles are charged with respect to all three forces. The lower two rows are the leptons. The third row, the neutrinos, carry only an isospin charge. The lower row, the charged leptons, also carry an electric charge. The right column lists the gauge bosons. The photon is an excitation of the electromagnetic field, the Z^0 and W^{\pm} are the carriers of the weak interaction. The eight gluons carry the strong force (image from Fermilabs, 1995)

3.2.1 Instantons

Using perturbation theory, it is not possible to write down interactions that are non-perturbatively in nature. Such interactions that do not have an infinitesimal form. An example of such interactions is a change of topology of the field. Solutions to such interactions are solitons, particle-like solutions linking ground states of different topology. As an example, consider a map of U(1) to U(1). This map has a non-trivial topology

$$e^{i\phi} \to e^{ni\phi}$$
 (3.13)

with *n* the winding number. This number arises because the periodicity allows non-trivial equivalence maps.
The instanton interaction arises because there does not exist a trivial map between the QCD symmetry group SU(3) and the S(3) symmetry of Wick-rotated spacetime at infinity (equation 3.15). As this map is not only fixed in space but also in time, it represents a soliton solution fixed in time, hence the name instanton. This instanton looks very much like an interaction. This is in contrast with the topological solution resulting from equation 3.13. Such a topological solution gives rise to soliton solutions, particle like solutions that carry the topological charge needed to transform any state to the other.

It is possible to write down this topological charge in terms of the fields. The equivalence for the winding number is the Pontryagin index q_T (Kapusta, 1989, ch. 8.4)

$$q_T = \frac{1}{16\pi^2} \int d^4 x \operatorname{Tr}[\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} F^{\mu\nu}]$$
(3.14)

This is a gauge invariant quantity, it is therefore not possible to get rid of this Pontryagin index by a gauge transformation. Using this equation it is possible to derive an interaction strength of this instanton interaction for particles carrying a colour charge. It can be shown that the strength inversely depends on the mass of the light quarks. Note that this topological charge is defined in Euclidean time, related to Minkowski time by a Wick rotation

$$t_e = it \tag{3.15}$$

The effects of QCD can only be accurately calculated at high energies, where perturbative QCD is applicable. In this domain the instanton effects are always dominated by perturbative corrections (Kapusta, 1989, ch. 8.4) and are therefore ignored in this thesis.

3.3 Confinement and asymptotic freedom

The fact that the weak and strong interactions are not Abelian has important consequences. In electromagnetism, vacuum polarisation shields the charged particles, leading to a smaller coupling constant for larger distances or lower energies. The energy dependence of the coupling constant is known as the running of the coupling constant. However, due to the self-interaction of non-Abelian theories, the coupling constant of non-Abelian theories behaves differently: it increases for larger distances! For the weak interaction the mass of the weak gauge bosons limits this effect, as a mass of the gauge boson gives rise to an exponential reduction of interaction strength at large distances. However, the strong interaction is exchanged by massless gauge bosons (gluons), so the inverse relation between coupling strength and energy is fully present in QCD.

Theories having this effect are called 'asymptotically free', as in the limit of infinite energy the interaction strength goes to zero. However, as the interaction energy drops, the coupling constant grows to high values. This suggests the existence of "confinement", the fact that at normal energies quarks cannot exist as individual particles. There is convincing evidence that in the limit of low energy, the field strength is independent of separation distance between the quarks. This leads to a spring-like force, where energy and separation distance depend linearly on each other. Trying to break a $q\bar{q}$ -pair by pulling them apart will increase the energy stored in the field, up to the moment where it is energetically favourable to create a new $q\bar{q}$ -pair form the vacuum, leaving two $q\bar{q}$ -pairs.

For this reason, at low energies quarks cannot exist as free particles. This is why in everyday life we see the world around us being build from mesons and baryons, particles made from respectively two or three quarks. These particles have no net colour charge (they are "white"). Mesons are thus build up from a quark and antiquark from the same colour-anticolour, while baryons must consist of three quarks with each a different colour charge (red, green or blue). The residual force of these quarks provides for the nuclear binding between hadrons, just as residual electromagnetic forces in molecular binding explains the Van der Waals binding. The Van der Waals-force results from the spatial distribution of electric charges in atoms and molecules. Close to the atom this can provide weak minima of the potential, allowing molecular binding at low temperatures.

The baryon binding is best described by pion exchange. Pions are massive mesons, carrying the quantum numbers of a quark-antiquark pair. Forces mediated by a massive particle fall off exponentially, as described by the Yukawa-potential

$$V(r) = -g^2 \frac{e^{-mr}}{r}$$
(3.16)

Using this potential, it is in principle possible to write down an equation of state for a baryonic compact star. However, especially in dense baryonic matter there are many uncertainties as a consequence of the renormalisation procedure, as well as the need to include many terms as V(r) becomes very large for small r. As a result there are many different models for the baryonic potential (see Sedrakian and Clark 2006 for an overview).

However, there exist situations where energies are not so low. One example is the early universe, where temperatures and energies increase as the size of the universe decreases. This high temperature-low baryon density limit can also be studied in particle accelerators, where the phase transition from nucleons to the quark-gluon plasma has possibly been observed. Temperatures where deconfinement occurs are of the order of 10^{12} K.

Another limiting case is the case of low temperature-high density. As densities grow, the separation distance between particles decreases and due to the exclusion principle the average particle energy increases. This leads to the deconfinement of the nucleons and the possibility of quark matter. The only possible objects with densities high enough to make such matter possible are compact stars. It might be energetically favourable to form strange quarks at increasing chemical potential, but before before confinement is lifted. If this is the case, the presence of baryons containing strange quarks, hyperons, can influence the equation of state.

3.3.1 Quantum chromodynamics at low energies

Since QCD is a strongly interacting theory at low energies, perturbative methods are inappropriate for these energies. Therefore, other methods are required to do calculations at these energies. One of the options is to tackle the problem of QCD in a non-perturbative way. This is usually done by a method known as lattice QCD. Here, spacetime is considered to be a discrete medium. This approach transforms integrals over all spacetime to discrete sums, which can be handled by computers. Of course the discretisation of space-time destroys translation symmetries. This can be solved by, after performing the integrals in the discrete algorithm, going back to a continuous spacetime again.

However, it is very hard to do lattice calculations at finite density. The chemical potential will insert imaginary exponents leading to fluctuations that are currently not under control. Therefore, this algorithm has up to now not resulted in usable calculations at high density. According to Ivanov et al. (2005) it is possible to do lattice calculations based on QCD inspired models in which this problem is under control. According to the authors the results rule out the existence of hybrid stars.

Another approach to tackle strongly interacting QCD is by the MIT bag model. Originally invented as a phenomenological model for protons, this model is also used for quark stars. The model is based on the idea that in strongly coupled QCD the vacuum will be filled virtual quark-antiquark pairs. These virtual pairs will produce a pressure, the bag pressure, confining the quarks in a cavity,

the bag. The pressure can in principle be calculated from QCD, but is mostly fitted to a desired size of the bag. Generally accepted values for the bag pressure are around 150 MeV (DeGrand et al., 1975).

This bag approach is also used for quark stars. This clearly demonstrates that a quark star is merely a huge baryonic particle, just as a neutron star is comparable to a huge nucleus. The disadvantage of this approach is that it is highly dependent on the chosen bag pressure, which cannot be calculated at finite density and energies below the scale where perturbative QCD becomes valid. At energies where perturbative QCD does become valid, it is probably preferable to use that technique, as it *is* QCD.

A third approach is the NJL model by Nambu and Jona-Lasinio (1961). This model describes the quark interaction as a four-fermion interaction. The gluon mediating this interaction is then contracted to a point. All the properties of the gluon interaction must then be described by the coupling constant of the NJL interaction. The advantage of this theory is that it provides simplifications allowing calculations at lower energies than perturbative QCD. The disadvantages are twofold. First, it is a non-renormalisable theory, allowing calculations only to a momentum cutoff. It can therefore not be used to probe the high energy limit of QCD. It is thought that quark matter in quark stars lies below this limit (Buballa, 2005). A second drawback is that it does depend on the phenomenological interaction strength of four quarks, something that is determined from the microscopic interaction including gluons that actually takes place. This theory only provides a phenomenological description of QCD, that still has to be fitted to the actual theory. It is very usable as a mechanism to calculate effects such as colour superconductivity (see section 6.2).

3.3.2 Phase transitions in dense matter



Figure 3.2: Strangeness as a function of chemical potential. Assumed is a mass difference between a neutron and a hyperon of 200 MeV. Right of the vertical line deconfinement is assumed, and again a mass of the strange quark of 200 MeV is assumed. This diagram is calculated assuming non-interacting particles, and a non-running strange mass. It is therefore not very accurate, and only for illustration.

Going from nuclear matter to quark matter, a phase transition has to occur where chiral symmetry is restored and confinement ceases to exist. Witten (1984) proposed the possibility that this actually might be the lowest energy state of matter at all densities. There is a potential barrier blocking the conversion of baryonic matter to quark matter. If matter becomes dense enough to overcome this barrier, the exothermic energy of the reaction then ensures the conversion of all matter to quark matter. Depending on the actual state of the matter, crusts of normal matter might still be possible (Alford et al., 2006b, Jaikumar et al., 2006b, Stejner and Madsen, 2005), while the effects of pure quark matter on the vacuum might seriously alter the spectrum of objects made of quark matter (Usov, 2001).

Less extreme is the idea that there exists a region of supernuclear density where quark matter is more stable than nuclear matter. It is then possible to convert only part of the compact object into quark matter. This leads to the possibility of

so-called hybrid stars, consisting of a quark matter inner core, a possible superfluid neutron matter outer core, and all the rest of the layers expected in neutron stars. These stars are hard to distinguish

Chapter 3: Quantum chromodynamics at high densities

from ordinary neutron stars. However, there are some options. The presence of quark matter inside alters the mass-radius relation, and as the equation of state differs properties may vary. Expect different cooling curves and a different behaviour of the magnetic field.

Another possibility is the formation of strange quarks before deconfinement occurs. The mass of the lightest hyperon, the Λ -particle consisting of of *uds* quarks, is about 1150 MeV, 200 MeV more than the mass of protons and neutrons. For chemical potentials > 200 MeV the exclusion principle favours the formation of hyperons. At this chemical potential quarks are still confined but the strangeness of the matter already increases. In figure 3.2 a plot of the situation is made.

The nature of the phase transition is still an open question. It depends on the coupling strength of the strong force, which is still poorly known at high densities. Around the transition, the strong coupling constant will be of order 1, the order where perturbation theory will fail to provide answers. There exist phenomenological theories, designed to describe features of QCD, which can be used to study this regime. However, as these theories are *not* QCD, the validity of the results is uncertain.

Another approach is to consider spacetime as a discrete medium, a lattice. This turns the path integrals over spacetime into sums which can be calculated. The advantage of lattice QCD, as this approach is known as, is that it automatically includes momentum cut-offs providing regularisation. Lattice QCD has proven useful in low energy calculations of QCD as well as high temperature calculations, but suffers from problems with high density calculations. These difficulties are known as the fermion sign problem, the result of the partition function acquiring an imaginary part.

3.4 The running coupling constant in QCD



Figure 3.3: Loop corrections to the gauge boson propagator are one of the contributions that lead to an energy dependent coupling constant

At high energies, the coupling constant decreases to values low enough for a truncated series expansion to become a good approximation. The series expansion can be determined from the action using the path-integral method. In (Freedman and McLerran (1977a) the propagators and vertices are calculated, in Freedman and McLerran (1977c) the thermodynamic potential is calculated. In these articles, the MOM renormalisation is used. To make gauge invariance manifest it is useful to rewrite this to the \overline{MS} scheme. Another reason to use the \overline{MS} is that most QCD-parameters have been calculated in the scheme.

One of the main problems of perturbative QCD is the renormalisation procedure. In QED, the renormalisation of the boson propagator is defined as follows:

$$\Pi(q^2)_{\text{renormalised}} = \Pi(q^2) - \Pi(0) \tag{3.17}$$

Here $\Pi(q^2)$ is the full propagator. Defining it in this way, renormalisation is uniquely defined by

the requirement that electromagnetic cross sections to drop to zero at zero momentum transfer. Via similar methods renormalisation uniquely defines the charge and mass in electrodynamics.

In contrast to electrodynamics, in QCD $\Pi(0)$ is not defined. Therefore we are unable to uniquely define the renormalisation procedure. The best we can do is to scale the running of the coupling constant to an experimentally defined amplitude. However, the exact procedure is not well defined, resulting in different outcomes for calculations in different schemes. Only when all terms in the series expansion are included the scheme dependence will drop out.

3.4.1 The running of the coupling constant

A very powerful way to study the running of the coupling constant $\alpha(\Lambda)$ is via the renormalisation group. The renormalisation group equation

$$M\frac{\partial g}{\partial M} = \beta(g) \tag{3.18}$$

where β is, to lowest order,

$$\beta_0(g) = -\frac{g^3}{48\pi^2} (11N - 2N_f) \tag{3.19}$$

expresses the evolution of the coupling constant in a differential equation. As can be seen from this equation, asymptotic freedom as discussed in section 3.3 occurs as long as β is negative. This is the case when $N_f < 5.5N$. Here N is the number of colours and N_f the number of flavours. The equation has the solution

$$\alpha(\Lambda) = \frac{g^2}{4\pi} = \frac{6\pi}{(11N - 2N_f)\log(\Lambda/\Lambda_0)}$$
(3.20)

Using experimental data, Λ_0 can be determined.

Yet, an uncertainty remains as α is specified as a function of Λ , the energy of in the gluon propagator carrying the energy $q^2 = \Lambda^2$. However, this is not a useful quantity, as we are unable to determine this energy. The relevant energy scale in our case will be the chemical potential μ . This can be solved by replacing $\Lambda \rightarrow c\mu$, with *c* a number of order unity. This introduces a new scale dependence into the equations. This parameter is the result of truncating the perturbation series expansion, and the scale dependence can be understood as a result of not describing the microscopic physics accurately. More on this in Kapusta (1989, ch. 4).

3.4.2 The running of the quantum chromodynamics coupling constant

Using the methods described in the previous section we can derive an equation for the coupling constant. Using not only β_0 , but also β_1 and β_2 we can determine α_s to three loop order (Particle Data Group, 2004)

$$\alpha_s = \frac{4\pi}{\beta_0 u} \left[1 - \frac{2\beta_1}{\beta_0^2} \frac{\log(u)}{u} + \frac{4\beta_1^2}{\beta_0^4 u^2} \left(\left(\log(u) - \frac{1}{2} \right)^2 + \frac{\beta_2 \beta_0}{8\beta_1^2} - \frac{5}{4} \right) \right]$$
(3.21)

In this equation $\beta_0 = 11 - 2N_f/3$, $\beta_1 = 51 - 19N_f/3$ and $\beta_2 = 2857 - 5033N_f/9 + 325N_f^2/27$, with N_f the number of flavours. Furthermore, $u = \log(\bar{\Lambda}/\Lambda_{\overline{MS}})$. $\Lambda_{\overline{MS}}$ can be determined using experimental data. Requiring $\alpha_s = 0.3089$ at $\bar{\Lambda} = 2$ GeV, $\Lambda_{\overline{MS}} = 365$ MeV for $N_f = 3$ (Particle Data Group, 2004), in the \overline{MS} scheme, as a determination of Λ is approximation scheme dependent.

Having defined α_s this way, the only uncertainty left is the dependence of $\overline{\Lambda}$ on μ . We take the dependence of Λ on μ to be $\Lambda(\mu) = c\mu$, as described in the previous section. The running of the coupling is plotted in figure 3.4.



Figure 3.4: *Running coupling constant to three loop order (equation 3.21)*

CHAPTER 4

Thermodynamics of degenerate quark matter

A typical star consists of about 10^{60} particles. For so many particles, a statistical approach is the only way to describe the system. Therefore, a thermodynamic approach to QCD is necessary. In this chapter we will first look at the general properties of the Fermi-Dirac distribution describing non-interacting Fermi-particles. However, we will later see that the interaction plays a crucial role in the realistic description of Fermi-matter in a star.

4.1 Fermi-Dirac distribution

One key property of fermions is that their wave function has to be fully antisymmetric. No fermion may have the same set of quantum numbers as any other fermion in the system. This property is known as Pauli's exclusion principle.

Using this principle, one can write down a density of states for a non-interacting Fermi gas. We assume a harmonic oscillator potential well, so all states are separated by an energy of $\hbar\omega$. Each individual state can only hold one fermion. The occupation number of such a state is thus 1 or 0. This also implies that in the ground state the fermion with the highest energy must have an energy of $(n + 1/2)\hbar\omega$, $n \in \mathbb{N}$. In case of high temperature, when the average fermion energy is much higher than that, the exclusion principle is not a severe limitation. In such cases there are much more states available than there are fermions, and the distribution will approach the Maxwell-Boltzmann distribution. However, in dense materials at low temperatures this limitation is very important. In everyday materials such as iron at room temperature the electrons in the iron have average energies equivalent to temperatures of order 5×10^5 K. In these cases the electrons will occupy all the low energy states, with only a few empty states near the Fermi surface at $(n + 1/2)\hbar\omega$. The partition function for one fermion energy level is

$$Z_{\text{free}}(\epsilon_i) = \sum_{n_s=0}^{1} e^{-\beta n_s(\epsilon_i - \mu)} = 1 + e^{-\beta(\epsilon_i - \mu)}$$

$$\tag{4.1}$$

From this the Fermi Dirac distribution can be obtained. It is

$$n(\epsilon_i) = \frac{1}{\exp[\beta(\epsilon_i - \mu)] - 1}$$
(4.2)

In these equations ϵ_i is the energy of a level, μ is the chemical potential and β the inverse temperature. In the second equation the energy is used to label the different states. Note that as $T \to 0$ this relation approaches the step function

$$n(\epsilon_i) = \Theta(\mu - \epsilon_i)$$

with $\Theta(x)$ is 1 for x > 0 and zero otherwise. From the partition function thermodynamic quantities can be derived. When considering a macroscopic object, the size of the object is much larger than all other physical length scales. Therefore, we can convert the sums over all states to integrals. It is convenient to label these states by their Fermi momentum. Using this one obtains

$$\epsilon = \int \frac{\mathrm{d}^3 p}{2\pi^3} \frac{\epsilon_i}{\exp[\beta(\epsilon_i - \mu)] - 1}$$
(4.3)

$$n = \int \frac{\mathrm{d}^3 p}{2\pi^3} \frac{1}{\exp[\beta(\epsilon_i - \mu)] - 1}$$
(4.4)

$$P = T \log\left[\int \frac{\mathrm{d}^3 p}{2\pi^3} \frac{1}{\exp[\beta(\epsilon_i - \mu)] - 1}\right]$$
(4.5)

These integrals can be converted to integrals over ϵ when the density of states is known. For an ideal relativistic gas the density of states is given by $\epsilon^2 = (p^2 + m^2)$.

At low temperatures, there is a sharp transition at $\epsilon = \mu$ from almost all states being occupied to all states being unoccupied. For a three-dimensional system there exists a three-dimensional volume in phase-space in which all the occupied states of the system lie. The volume is bounded by the Fermi surface. In case of a free Fermi gas this phase space volume will be one eighth of a ball, in more general models the shape of the volume can be altered. However, the existence of a sharp transition surface is a general feature of all Fermi systems at low temperatures.

For compact objects this is a fundamental concept. When a star runs out of nuclear fuel, there is no energy generating mechanism available to provide a thermal gradient and thus a pressure gradient to counter gravity. As such a star radiates away energy, it has to respond by contracting. This process continues until there is a new pressure generating mechanism available to sustain gravity. In white dwarfs this is the degenerate pressure of electrons, while in neutron stars degenerate neutrons and in quark stars degenerate quarks deliver the pressure.

As mentioned in section 2.1, degeneracy can lift the Fermi level to relativistic energies. In case of this relativistic degeneracy, the solution of the structure equations becomes insensitive to central pressure and radius, indicating a maximum mass, the Chandrasekhar mass (see equa-



Figure 4.1: *Running mass of the strange quark, calculated to one-loop order in alpha (the first two terms of equation 3.21)*

tion 2.23). Relativistic degenerate means that the mean kinetic energy per fermion is more than the rest mass of that fermion. For a white dwarf to become relativistic degenerate the average fermion energy thus has to be larger than 0.511 MeV.

In quark star models the chemical potential at the centre is of order 500 MeV, while up- and downquark masses are of order 5 MeV. Although the strange quark is much heavier, its energy dependent mass is still well below this limit, for an assumed dependence $\Lambda(\mu) = c\mu$, $c \sim 2$ (see section 3.4 for a discussion). Therefore, quark star matter can be approximated by an equation of state for relativistic degenerate fermions. In order to understand why solutions to the structure equations using a quark equation of state are dependent on central pressure, we need to include interactions. To do this we will first use a phenomenological model, known as the MIT bag model, to do some qualitative study. Later we will derive interaction models based on perturbative QCD.

4.2 The MIT bag model

In the MIT bag model quarks are considered to be free particles confined to a bounded region by a bag pressure *B*. This extra pressure term is a phenomenological description of the quark-quark interaction binding the quarks. A convenient way to describe the thermodynamics of a system is to use the grand thermodynamic potential $\Omega = -U - \mu n - Ts$, with *U* internal energy, *n* particle density and *s* entropy density. Furthermore, μ is the chemical potential and *T* the temperature. From this potential quantities as pressure, number density and energy density can easily be derived. They are

$$P(\mu) = -\Omega(\mu) \tag{4.6}$$

$$n(\mu) = \frac{\partial p}{\partial \mu} \tag{4.7}$$

$$\epsilon(\mu) = -p(\mu) + \mu n(\mu) \tag{4.8}$$

For the MIT bag model the thermodynamic potential becomes

$$\Omega(\mu) = -\frac{N_f \mu^4}{4\pi^2} + B = -\frac{1}{3}(\epsilon - 4B)$$
(4.9)

The first part is the thermodynamic potential of a free Fermi gas, the second term is the bag pressure. Accepted values of this bag pressure are of order 150 MeV (DeGrand et al., 1975).

As can be seen, the effect of the bag constant is to generate a vacuum pressure, making it possible for the applied pressure to become zero before the energy density drops to zero. This in contrary to a free gas where $\epsilon(P = 0) = 0$. Using the bag model as an equation of state in the structure equations makes it possible to find solutions which depend on central pressure and are thus mass dependent. This is discussed in more detail in section 5.1.1.

4.3 Finite temperature field theory

In the previous chapter zero-temperature, zero density QCD was discussed. However, we need a description of QCD applicable in dense matter. This requires a statistical approach. We will again use the partition function

$$Z = \operatorname{Tr} \exp\left[-\beta(H - \mu_i N_i)\right] = \int \mathcal{D}\phi \left\langle \phi \right| e^{-\beta(H - \mu_i N_i)} \left|\phi\right\rangle$$
(4.10)

where \mathcal{D} denotes a functional integration (see appendix B). In this equation $\beta = (T)^{-1}$ and *H* is the Hamiltonian of the system, as usual. By a Laplace transform this has the particle number operator N_i as a conserved charge with the chemical potential μ as the Lagrange multiplier. The role of β is similar: it is the Lagrange multiplier setting the mean energy of the system, with *H* as the conserved energy.

From the partition function various thermodynamic quantities can be derived

$$P = T \frac{\partial \log Z}{\partial V} \tag{4.11}$$

$$N_i = T \frac{\partial \log Z}{\partial \mu_i} \tag{4.12}$$

$$S = \frac{\partial T \log Z}{\partial T} \tag{4.13}$$

$$E = -PV + TS + \mu_i N_i \tag{4.14}$$

The partition function, equation 4.10, can be used to write down the partition function for an ensemble of particles. In this example I will discuss bosons. The expectation value for $\exp[-iHt]$ is

$$\langle \phi | e^{-iHt} | \phi \rangle = \int \mathcal{D}\pi \int_{\phi} \mathcal{D}\phi \exp\left[i \int_{0}^{t} dt \int d^{3}x \left(\pi(\mathbf{x}, t) \frac{\partial \phi(\mathbf{x}, t)}{\partial t} - \mathcal{H}(\pi(\mathbf{x}, t), \phi(\mathbf{x}, t)) \right) \right]$$
(4.15)

which can be derived using the path integral method (see Kapusta, 1989). The trick is to use the similarity between the time translation operator $\exp(-iHt)$ and the partition function $\operatorname{Tr}[\exp(-\beta H)]$. First we replace the time coordinate with an imaginary one: $\tau = it$. Doing this in equation 4.15, it is clear that β has the effect of changing the integration range of *t*. When $T \downarrow 0 \Rightarrow \beta \rightarrow \infty$, so in that case the integration range for τ is $[0, \infty)$. For $T \neq 0$ the integration range for τ changes to $[0,\beta]$, whilst requiring that $\phi(\mathbf{x}, 0) = \phi(\mathbf{x}, \beta)$, a consequence of taking the trace.

Using all this, the partition function for bosons becomes

$$Z = \int \mathcal{D}\pi \int_{\text{periodic}} \mathcal{D}\phi \exp\left[\int_{0}^{\beta} d\tau \int d^{3}x \left(i\pi \frac{\partial \phi}{\partial \tau} + \mathcal{H}(\pi, \phi) + \mu \mathcal{N}(\pi, \phi)\right)\right]$$
(4.16)

A similar technique can be applied for a fermion field, arriving at

$$Z = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left[\int_{0}^{\beta} i\mathrm{d}\tau \int \mathrm{d}^{3}x \,\overline{\psi}\left(-\gamma^{0}\frac{\partial}{\partial\tau} + i\gamma \cdot \nabla - m + \mu\gamma^{0}\right)\psi\right]$$
(4.17)

From these equations, one can evaluate the partition function and then determine the thermodynamic potential using perturbation theory. In section 4.3.2 we will see how to do this in the limit where $T = 0, \mu \neq 0$. First, let us look at some of the formalism required.

The partition function for an N particle system is

$$Z = N \int \mathcal{D}\phi e^{S(\phi)} \tag{4.18}$$

where the action $S(\phi)$ can be written as $S(\phi) = S_0(\phi) + S_I(\phi)$, with $S_0(\phi)$ the free part quadratic in the fields and $S_I(\phi)$ the interactions of higher order. Using this, equation 4.18 can be written as

$$Z = N \int \mathcal{D}\phi e^{S_0(\phi)} \sum_{l=0}^{\infty} \frac{1}{l!} S_I^l(\phi)$$
(4.19)

where $S_I^l(\phi)$ is the perturbative term of order *l* in the coupling constant. We can then take the logarithm of this equation

$$\log Z = \log\left(N\int \mathcal{D}\phi e^{S_0(\phi)}\right) + \log\left(1 + \sum_{l=1}^{\infty} \frac{1}{l!} \frac{\int \mathcal{D}\phi e^{S_0(\phi)} S_I^l(\phi)}{\int \mathcal{D}\phi e^{S_0(\phi)}}\right)$$
(4.20)

Note that this equation is now split in a free and an interacting part

$$\log Z = \log Z_0 + \log Z_I \tag{4.21}$$

The free part is the ideal gas contribution, as discussed in section 4.1. The interactions are written as a perturbation series expansion. The actual calculation for a term of order l in the coupling constant is then

$$\left\langle S_{I}^{l}(\phi)\right\rangle_{0} \equiv \frac{\int \mathcal{D}\phi e^{S_{0}(\phi)} S_{I}^{l}(\phi)}{\int \mathcal{D}\phi e^{S_{0}(\phi)}}$$
(4.22)

For a coupling constant $\alpha/\pi < 1$ the high order terms will be small, ideally going to zero as $l \to \infty$. This suggests that a good approximation can be obtained using only the terms $S_I^l(\phi)$ with $l < l_{\text{term}}$, where l_{term} is the limit of the expansion chosen such that the desired accuracy is achieved.

However, the number of possible interactions in $S_I^l(\phi)$ increases with l. For a series expansion of $S_I^l(\phi)$ the best that can be achieved is that the series expansion converges, the sum of the asymptotic series expansion generally does not. The best accuracy that can then be achieved is the series expansion up to $l_{\text{term}} \leq l_{\text{max}}$, where l_{max} is determined by the condition

$$\sum_{l=1}^{l=l_{\max}} \left\langle S_I^l(\phi) \right\rangle_0 < \sum_{l=1}^{l=l_{\max}+1} \left\langle S_I^l(\phi) \right\rangle_0 \tag{4.23}$$

As an extra limit to this is the computational effort required for higher order terms. These terms generally contain contributions of many different possible interactions, resulting in a limit that is in practice a far stronger limit than the limit in equation 4.23.

4.3.1 The thermodynamic potential

In thermodynamics, the thermodynamic potentials are the potentials of the different ensembles. In this thesis the ensemble used is the grand ensemble. Therefore the related thermodynamic potential, the grand potential, will be referred to as 'the thermodynamic potential'.

$$\Omega = U + Ts + \mu_i N_i = \beta^{-1} \log Z \tag{4.24}$$

Since $\Omega \propto \log Z$, the above equation can be handled perturbatively in the region of small coupling constant, as shown in equations 4.18-4.22. Since $\log Z$ is now described in terms of the action, standard field theoretical methods can be used. However, in a non-zero temperature or density, extra terms appear. This is the result of infrared divergences, which give zero at zero temperature or density, but yield a finite value if the temperature or chemical potential cannot taken to be zero (Kapusta, 1989, ch. 3)

When T = 0 but $\mu \neq 0$ terms of order $q^4 \log q^2$ will appear (Kapusta, 1989, p. 75). These terms are the effect of corrections to the gluon self energy interacting with a thermal or dense vacuum, resulting in an effective mass for the gluon. As the perturbation is done assuming zero gluon mass, extra perturbative terms compensating this arise. The dense interactions can be written as ring diagrams or plasmon corrections (Gell-Mann and Brueckner, 1957). These diagrams can be summed up to obtain a term that is not an integer power of α . This term has to be added to the series expansion obtained from equation 4.22.

This effect is present in QED, where the plasmon contribution is the same as in QCD. We will do the calculation for QED, which is much easier but leads to the same result (Kapusta, 1989, ch. 8).



Figure 4.2: *Ring diagrams, where* α_s *is the strong coupling constant.*

The contribution from the ring diagram can be written as

$$\frac{\log Z_{\rm ring}}{\beta V} = -\frac{1}{2}T \sum_{n} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{Tr} \left(\log[1 + \Pi_0(k)\Pi(k)] - \Pi_0(k)\Pi(k) \right)$$
(4.25)

with the photon propagator Π_0 given by

$$\Pi_0^{\mu\nu} = \frac{1}{G(k_\mu) - k^2} P_T^{\mu\nu} + \frac{1}{F(k_\mu) - k^2} P_L^{\mu\nu} + \frac{\rho}{k^2} \frac{k^\mu k^\nu}{k^2}$$
(4.26)

and $\Pi(k^2)$ given by

$$\Pi_{\mu\nu} = \Pi_{\mu\nu}^{-1} - \Pi_{0\mu\nu}^{-1} \tag{4.27}$$

with $\Pi_{\mu\nu}^{-1}$ the inverse of the full propagator. Furthermore, *F* and *G* are scalar functions of k^0 and $|\mathbf{k}|$, P_T and P_L are transversal and longitudinal projection operators and ρ is a gauge fixing parameter $k^{\mu}k^{\nu}\Pi_{\mu\nu} = \rho$. Using these equations equation 4.25 can be rewritten to

$$\frac{\log Z_{\text{ring}}}{\beta V} = -\frac{1}{2}T \sum_{n} \int \frac{\mathrm{d}^{3}k}{(2\pi)^{3}} \left(2\log\left[1 - \frac{G(n,\omega)}{k^{2}}\right] + 2\frac{G(n,\omega)}{k^{2}} + \log\left[1 - \frac{F(n,\omega)}{k^{2}}\right] + \frac{F(n,\omega)}{k^{2}} \right)$$
(4.28)

F and *G* are functions of *n* and ω as $k^0 = 2i\pi n\mu$ and $\omega = |k|$. This function is not finite, and to isolate the infinities you can calculate equation 4.28 at $\omega = n = 0$. The finite term is then the difference between equation 4.28 at $\omega = n = 0$ and $\omega, n \neq 0$. The resulting equation yields

$$\frac{\log Z_{\text{ring}}}{\beta V} = -\frac{1}{4}T \sum_{n \neq 0} \int \frac{d^3k}{(2\pi)^3} \left\{ \left[2\left(\frac{G(n,\omega)}{k^2}\right)^2 + \left(\frac{F(n,\omega)}{k^2}\right)^2 \right] + 2\left(\frac{G(0,\omega)}{\omega^2}\right)^2 - 2\left(\frac{G(0,0)}{\omega^2}\right)^2 + \left(\frac{F(0,\omega)}{k^2}\right)^2 - \left(\frac{F(0,0)}{\omega^2}\right)^2 \right\}$$
(4.29)

As T = 0 in our case, it is useful to perform a Wick rotation $k_4 = -ik_0$ and to introduce $k_E^2 = k_4^2 + \mathbf{k}^2 = -k^2 \ge 0$. Note that $\beta \propto T^{-1}$, so equation 4.29 does not yield 0. Since *F* and *G* are functions of k_{μ} , these functions now depend on the Euclidean coordinates. Independent coordinates are $|k_E|$ and ϕ with tan $\phi = |\mathbf{k}|/k_4$. Using all this we can transform the ring contribution to

$$\frac{\log Z_{\text{ring}}}{\beta V} = -\frac{1}{(2\pi)^3} \int_0^\infty dk_E^2 k_E^2 \int_0^{1/2\pi} d\phi \sin^2 \phi \left[2 \log \left(1 + \frac{G(k_E^2, \phi)}{k_E^2} \right) - 2 \frac{G(k_E^2, \phi)}{k_E^2} + \log \left(1 + \frac{F(k_E^2, \phi)}{k_E^2} \right) - \frac{F(k_E^2, \phi)}{k_E^2} \right]$$
(4.30)

When using the explicit forms of F and G (see Freedman and McLerran, 1977b). one can obtain a ring contribution in the relativistic limit

$$\frac{\log Z_{\rm ring}}{\beta V} = -\frac{e^4 \log e^2}{128\pi^6} \mu^4$$
(4.31)

As noted, in QCD this result is still valid and give rise to terms of order $g^4 \log g^2$ (see Freedman and McLerran, 1977c).

4.3.2 Thermodynamic quantities of a quark gas



Figure 4.3: Two particle interaction to second order with renormalised propagators



Figure 4.4: Three particle interactions with renormalised propagators

Using the methods described above it is possible to derive the thermodynamic potential for a gas of free quarks. This is done in Freedman and McLerran (1977c) and Baluni (1978) in the MOM subtraction scheme. The terms included are the ring diagrams as given above, the two-particle to twoparticle transition diagrams to second order in the strong coupling α_s and three-particle interactions. In figures 4.3-4.4 a schematic overview of the contributing interaction diagrams is provided. Not included in these figures are all permutations of the diagrams. Furthermore, we need to take the trace of the diagrams and integrate over all momenta. This as equation 4.22 expresses vacuum to vacuum expectation values, while the diagrams represent particle interactions.

The total thermodynamic potential is given by

$$\Omega = \Omega^{(0)} + \Omega^{(1)} + \Omega^{(2)} + O(\alpha_s^3)$$
(4.32)

with $\Omega^{(n)}$ the thermodynamic potential correction of order α_s^n . The zero and one loop correction

potential for both massless and massive quarks for $T = 0, \mu \neq 0$ is (Baluni, 1978)

$$\Omega^{(0)} = D_f \sum_a (-\frac{4}{3}\pi^2) \phi_0 \left(\frac{m_a^2}{\mu_a^2}\right) \left(\frac{\mu_a}{2\pi}\right)^4$$
(4.33)

$$\Omega^{(1)} = D_f C_f \alpha_s \sum_a (2\pi) \phi_2 \left(\frac{m_a^2}{\mu_a^2}\right) \left(\frac{\mu_a}{2\pi}\right)^4 \tag{4.34}$$

and ϕ_0 and ϕ_2 are given by

$$\phi_0(x) = (1-x)^{1/2}(1-\frac{5}{2}x) + \frac{3}{2}x^2 \log\left[\frac{1+(1-x)^{1/2}}{x^{1/2}}\right]$$
(4.35)

$$\phi_2(x) = 3\left\{ (1 - x^{1/2} - x \log\left[\frac{1 + (1 - x)^{1/2}}{x^{1/2}}\right] \right\} - 2(1 - x)^2$$
(4.36)

The second order term has only been calculated for a massless quark

$$\Omega^{(2)} = D_f C_f \beta^{(2)} \alpha_s \sum_a \log\left(\frac{2\mu_2}{eM}\right) \left(\frac{\mu_a}{2\pi}\right)^4 - D_f C_f \alpha_s^2 \sum_a \left\{\frac{5}{8}C_c + \left(C_f - \frac{1}{2}C_c\right) \left(\pi^2 + \frac{21}{4} - 165\right)\right\} \left(\frac{\mu_a}{2\pi}\right)^4$$
(4.37)

with

$$\beta^{(2)}(\alpha) = \left\{ -\frac{11}{33}C_c + \frac{4}{3}N_f D_f C_f / D_c \right\} \alpha^2$$
(4.38)

Since the second order contribution for massive quarks is unknown, we have to take $\Omega^{(2)} = 0$ for massive quark flavours.

In the above equations $C_{c,f}$ and $D_{c,f}$ are the Casimir numbers for respectively the colour and flavour symmetry group. $N_{c,f}$ are, respectively, the number of colours and flavours. More specific

$$2NC_f = D_c = N^2 - 1, \qquad C_c = D_f = N \tag{4.39}$$

In nature $N_c = 3$. The value of N_f depends on which quark flavours contribute to the diagrams. As the quarks all have different masses it can be seen as an energy dependent value. At low energies only two quarks will be present, so $N_f = 2$ in that case. In quark stars the chemical potential will be high enough to overcome the strange quark mass, so in quark stars $N_f = 3$, as forming strange quarks will lower the Fermi levels and thus be a state of lower energy. The charm, bottom and top quarks are much heavier than the up, down and strange so it is not energetically favourable to form these quarks.

The above formulae are in the momentum-space subtraction (MOM) scheme. Much more common is the $\overline{\text{MS}}$ scheme. This scheme has the advantage of preserving manifest gauge invariance. The relation between MOM and $\overline{\text{MS}}$ is

$$\frac{\alpha_s^{\text{MOM}}}{\pi} = \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \left[1 + \mathcal{A}\alpha_s^{\overline{\text{MS}}} \right]$$
(4.40)

with $\mathcal{A} = 151/48 - (5/18)N_f$. This leads to a thermodynamic potential to second order for massless quarks of

$$\Omega(\mu) = -\frac{N_f \mu^4}{4\pi^2} \left\{ 1 - 2\left(\frac{\alpha_s}{\pi}\right) - G + N_f \log\left(\frac{\alpha_s}{\pi}\right) + \left(11 - \frac{2}{3}N_f\right) \log\left(\frac{\bar{\Lambda}}{\mu}\right) \left(\frac{\alpha_s}{\pi}\right)^2 \right\}$$
(4.41)

with $G = G_0 - 0.536N_f + N_f \log N_f$ where $G_0 = 10.374 \pm 0.13^{-1}$. $\overline{\Lambda}$ is the renormalisation subtraction point.

It is also possible to include the nonzero strange quark mass. In that case the equation for the massive quark becomes, to first order (Fraga and Romatschke, 2005)

$$\Omega_{\text{mass.}} = -\frac{N_c}{12\pi^2} \left[\mu u \left(\mu^2 - \frac{5}{2} m^2 \right) + \frac{3}{2} m^4 \log \left(\frac{\mu + u}{m} \right) \right] + \frac{\alpha_s (N_c^2 - 1)}{16\pi^3} \left[3 \left(m^2 \log \left(\frac{\mu + u}{m} \right) - \mu u \right)^2 - 2\mu^4 + m^2 \left(6 \log \frac{\bar{\Lambda}}{m} + 4 \right) \left(\mu u - m^2 \log \left(\frac{\mu + u}{m} \right) \right) \right]$$
(4.42)

with $u = \sqrt{\mu^2 - m^2}$ and *m* the energy dependent running mass

$$m(\mu) = \hat{m} \left(\frac{\alpha_s}{\pi}\right)^{4/9} \left(1 + 0.895062 \frac{\alpha}{\pi}\right)$$
(4.43)

with $\hat{m} \simeq 262$ MeV. As *m* depends on α_s , the scale dependence of α_s is inherited by *m*. Note that, in the limit of $m \downarrow 0$, equation 4.42 reduces to the first two terms of equation 4.41, as it should.

¹The uncertainty in G_0 is from errors in numerical integration as performed by Freedman and McLerran (1977c)

CHAPTER 5

Mass-radius relation of quark stars

5.1 Numerical stellar models

As discussed in section 2.1, stellar models can be described using the following set of equations: an equation describing the pressure gradient given a certain distribution of matter and a set of equations describing what the pressure will be given a matter distribution. Together these equations can be integrated and yield a model of a star.

The first equation, that of how the matter is distributed given a thermal gradient, is called the equation of hydrostatic equilibrium. In this thesis we will use it in relativistic form, the TOV equation (equation 2.1). The set of equations describing the state of matter given an energy distribution is called the equation of state, an equation that relates pressure, energy density and temperature. For nonradiating objects these equations are enough to describe the system. For radiating objects like real life stars radiation pressure might be important in certain regions of the star, and must therefore also be included.

In the case of neutron stars and quark stars, the only form of radiation is due to cooling, and is only important for the stellar structure during and just after the collapse. During these moments, the star is very far from equilibrium, not justifying the use of the equation of hydrostatic equilibrium in the first place. These situations are therefore computationally very much more complex than the situation of a settled star in equilibrium.

But even in equilibrium, only the very simplified scenario using Newtonian gravity discussed in section 2.1.1 has analytic solutions. In general relativity it is not possible to analytically solve the set of equations 2.1 and 2.2 with the polytropic equation of state (equation 2.6). Yet including general relativity but leaving out other interactions is not good enough for neutron stars and quark stars. Dense matter in neutron stars and quark stars is characterised by strong internal interactions, so an ideal gas approach is insufficient. Instead of just guessing a good polytropic index γ it is therefore better to explicitly account for these interactions.

When calculating models of quark stars temperature can generally be ignored, as the binding energy is many orders of magnitude larger than the thermal energy. The remaining set of equations to be solved contains only two differential equations and one equation linking energy density to pressure, requiring two boundary values. Typical boundary values would be to require that $r(P = 0) = R_*$, where R_* would be a user-supplied value. Another natural boundary would be to replace the radius with the total mass of the star, still requiring the pressure to vanish at the surface. In the latter approach, however, the radius is not a pre-determined condition but is calculated by the vanishing-pressure requirement.

These approaches work in principle, but are computationally very difficult. The most straightforward choice for boundary conditions is to start calculating from an initial central pressure and keep iterating through the equations until the pressure is zero. Again $r(P = 0) = R_*$ and $\frac{4}{3\pi}R_*^3\bar{\rho} = M_*$, but both parameters have become internal variables. Calculating such models for a range of initial pressures produces outcomes of models with different masses and radii. These outcomes can then be combined in a mass-radius diagram.

Typically when the central pressure is increased, the mass will be increased. This can be understood as the central pressure is caused by the weight of all the mass on top of it. If mathematical solutions give other results, these results are not physical.

It is possible that for a given central pressure there are multiple solutions for central energy. White dwarf stars, for example, can have masses that are also possible for neutron stars. Unless of cosmological origin, nature will always favour the first stable solution, as all concentrations of mass have formed as a result the collapse of gas in the interstellar medium, a very dilute material. As knowledge of white dwarf structure is well-established, and the predictions on the maximum mass of these objects are all of the order of the Chandrasekhar limit, neutron stars or quark stars much lighter than the Chandrasekhar limit will be considered as unphysical. Note that rotation provides an extra balancing force against pressure and is thus not a mechanism of creating underweight neutron stars or quark stars.

The maximum mass of an object can be found using a plot of the mass radius diagram. Most of these diagrams will show a peak in mass at a certain radius, allowing solutions for all masses smaller than M_{max} . Increasing the central density further after M_{max} leads again to smaller masses. This is a signature of reducing mass while increasing the central pressure and is unphysical. All solutions beyond $R_*(M_{\text{max}})$ where the tangent vector to the mass-radius plot is negative should therefore be seen as a mathematical rather than a physical solution.

5.1.1 The MIT bag model

As discussed in section 2.1.1 it is possible to write the interactions as an extra energy term contributing to the internal energy. Using the MIT bag pressure (section 4.2) as the extra energy term, it is possible to study the general behaviour of massless quark star models. The MIT bag model will provide a vacuum pressure B that makes it possible that solutions to equations 2.1 and 2.2 span a range of masses, contrary to the solution in equation 2.23, which uniquely defines a maximum mass.

The internal energy is changed by the presence of a bag pressure by

$$\Delta E_{\rm MIT} = -\frac{4B}{3n(\epsilon)} \tag{5.1}$$

indicating that when P = 0, there is still an energy density of 4/3B. This has the effect of binding the star tighter together, just as a general relativistic correction does. However, the change in energy ΔE_{MIT} is independent of the density and radius, in contrast with ΔE_{GR} , which does depend on both density and radius.

Combining the above equation with equations 2.24 and 2.25 the equation for energy balance becomes

$$E = \frac{N^{1/3}}{r} - \frac{GN\epsilon_p^2}{r} - 0.918294M^{7/3}\rho_c^{2/3} - \frac{4B}{3n(\epsilon_p)} = 0$$
(5.2)

Here the Newtonian gravitational potential is somewhat strange, as massless particles are assumed. Using $E = mc^2$ it is possible to use the particle energy as its mass. Furthermore, the equation for the general relativistic correction was derived using a non-interacting gas, so the term will differ from the term in this equation, as discussed below equation 2.25. I have used the energy per particle as parameter. The energy per particle is

$$\epsilon_p = \frac{3\mu}{4} + \frac{B\pi^2}{\mu^3 N_f} \tag{5.3}$$

Solving equation 5.2 for r yields

$$r \approx \frac{\sqrt[3]{N} - GN\epsilon_p^2}{0.92\rho_c^{2/3}M^{7/3} + \frac{4B}{3n(\epsilon_p)}}$$
(5.4)

It is clear that both the general relativistic correction and the bag pressure change the situation of section 2.1.1, where no solution existed for a relativistic gas. The effect of general relativity is to decrease the maximum stable mass somewhat. This can be understood in that general relativity increases the potential, so a collapse will occur earlier. The bag pressure also allows a smaller maximum mass and radius, for the same reason as with general relativity. So models with interaction generally lead to solutions with a smaller radius than models without any interaction.

It is interesting to note that the bag pressure does not depend on any stellar parameter. For a very light star, the bag pressure is the same as for a more massive star. For models of a light star, the bag pressure is then the dominant contribution, so we can ignore the gravitational potential

$$r = \frac{3\sqrt[3]{Nn(\epsilon_p)}}{4B}$$
(5.5)

This equation scales as $r \propto N^{1/3}$, the scaling law of the mass-radius equation for an incompressible medium. The presence of the particle density is not important for small objects as the particle density barely changes over the object.

For model calculations equation 5.2 for energy density instead of energy per particle is probably more useful. From this equation it is clear that in the low mass scenario the energy density is independent of the radius, a signature of an incompressible medium.

$$E = \epsilon_{\rm free} - \frac{4}{3}B - \frac{GM(r)\epsilon}{r} - 0.918294M^{7/3}\rho_c^{2/3} = 0$$
(5.6)

This equation uses again a non-relativistic gravity with a correction term for models using an ideal gas equation of state.

Only at very high mass general relativity starts to play a role. It is quite easily seen that general relativity steepens the gradient in the mass-radius plot. At high masses the numerator of equation 5.4 decreases due to the term $\propto N\epsilon_f^2 = M\epsilon_f$ while the denominator increases as *M* increases. As expected, both the higher energy density and the greater mass contribute to the steepening of the potential. The general relativistic correction term assumes an ideal gas equation of state. Extra attractive interactions result in smaller stars, increasing the effect of general relativity. This can only be calculated using a general relativistic equation for gravity.

The change to a polytropic equation is that the pressure vanishes at finite energy density. In other words, the polytropic equation (equation 2.6) for a relativistic degenerate gas changes to

$$P = \frac{1}{3}(\epsilon - 4B) \tag{5.7}$$

where ϵ is the energy density without the energy density from interactions. Also from this equation it can be understood that the Lane-Emden equation (equation 2.19) has more than one solution for a relativistic gas when a bag pressure is included. Equation 5.7 can be rewritten to

$$P - \frac{4}{3}B = \frac{1}{3}\epsilon \tag{5.8}$$

indicating that the bag pressure can be seen as an alteration of the boundary condition. For an ideal relativistic gas the boundary condition is P = 0, while for an interacting gas with a phenomenological bag pressure as interaction term the boundary condition changes to P = 4/3B. While for a relativistic gas $P \downarrow 0$ only as $r \rightarrow \infty$, r remains finite when a nonzero boundary condition P can be used.

5.2 Numerically solving models of quark stars

5.2.1 The method

As the TOV equations combined with a quark matter equation of state does not have an analytical solution, I have used a computer programme to numerically solve the set of equations. The program is written in C, and uses the Bulirsch and Stoer method described in Numerical Recipes in C (Press et al., 1996) to integrate the set of differential equations. This method uses a rational function extrapolation to describe the equations and integrates these. When combined with an adaptive step size method based on a required precision this model produced accurate results in a reasonable time span (see figures 5.9 and 5.11).

The integration sequence starts with an initial pressure, or chemical potential which I used as running variable. Using this chemical potential the pressure and energy density are calculated. These results are fed in the TOV equation, resulting in a pressure gradient. Using the relation

$$\frac{\mathrm{d}\mu}{\mathrm{d}r} = \frac{\mathrm{d}P}{\mathrm{d}r}\frac{\mathrm{d}\mu}{\mathrm{d}P} \tag{5.9}$$

the pressure gradient can then be converted back into a gradient in chemical potential.

Using this gradient, a next step δx from the origin can be calculated. The size of δx is determined by the slope of the gradient: the steeper the gradient the smaller the step size. This algorithm not just allows for a good precision and relatively low computing power, it also guarantees that the precision is constant throughout the model.

This process is repeated until $P(\mu) = 0$. The mass and radius are then given by

$$R_* = \sum_{\text{all steps}} \delta x \tag{5.10}$$

$$M_* = 4\pi r^2 m(r)$$
(5.11)

which are both easily obtained from the calculations: $r = \sum \delta x$ is the running parameter used to iterate through the star, M is the solution of one of the equations of the model, the equation of mass conservation, at $r = R_*$.

When this process is repeated for a range of initial chemical potentials a mass-radius relation is obtained. I have used as a lower limit for the central chemical potential $\mu_c = \mu(p = 0)$, for which one expects a star with zero radius. The higher limit I have set by hand, such that the the most massive solution of the model is included.



5.2.2 Equations of state used in the model

Figure 5.1: Results of compact star models using an equation of state of quark matter matched to hadronic matter. The mass-radius for quark star models are the same as in section 5.4.1. In this plot $\rho_0 \sim 0.48 \text{ fm}^{-3}$ is quark number density at nuclear density (image from Fraga, 2006).

Using the method described above, the physics lies in the TOV equations and the equations of state used. I have based the calculations on the assumption that quark matter does exist and makes up the whole star. I have not included the possibility of a hybrid star, where a normal neutron exterior would be accompanied by a quark matter core. Although it is rather straightforward to calculate mass-radius relations of hybrid stars, I don't consider it very useful as both the quark equation of state and the high density neutron equation of state are poorly constrained. Models of both quark stars and neutron stars are strongly dependent on the equation of state. Combining the two only introduces as an extra uncertainty the pressure at which confinement occurs and thus increases the degeneracy of the model. Further study to the equation of state of both quark matter and neutron matter would be necessary to improve this situation.

The quark equation of state used is based on a perturbative calculation of the thermodynamic potential of a quark gas at zero temperature. The

thermodynamic potential uniquely relates P and ρ , and is therefore very suitable as an equation of state. In the previous chapters it is described how to calculate such a thermodynamic potential. The thermodynamic potential is calculated to second order in α_s for massless, and to first order in α_s for any number of massive quarks. I have performed model calculations up to second order, and also studied the effect of one massive quark species, up to first order.

As discussed in section 3.3.1, it is also possible to study quark matter using a phenomenological model. Two often used models are the MIT bag model and the NJL model. The MIT bag model simulates confinement by the introduction of a vacuum pressure, the bag constant, that act as a force on the quarks confining them to a region in space. This has the effect of shifting the line in the pressure-density diagram such that at zero pressure there is still a finite energy density. As can be seen in figure 5.4 this is not very far from the truth. However, the bag pressure is not determined by the model and has to be fitted to observations. As there currently do not exist observations of very dense quark matter, it is difficult to establish proper bounds on the bag pressure. The bounds currently used are based mainly on stability criteria for normal nuclei and input from other methods.

The NJL-model is an effective field theory describing the quark interaction as a four-fermion interaction. Since there is no way to derive the interaction strength of the four-point interaction from the theory, and introduces another free parameter. Also for this model the free parameters can be fitted to the hadron spectrum.

5.2.3 The programme

The code of my programme can be found at http://www.science.uva.nl/~srhardem/code. html. The programmes work by calculating a number of models from a lower chemical potential to an upper one. Between these bounds the programme calculates a logarithmic distribution as that guarantees an even spread of points in the mass-radius relation.

For each model, the method described above is used to calculate the pressure and density, which are an input for the TOV equation. This process is iterated until P = 0. For the model including a massive quark species, somewhat more work had to be done. As one quark gets a mass, the quark matter is no longer strictly neutral by itself. This has to be compensated by adding an electron-content. Also, there are now two different chemical potentials. All in all, charge neutrality requires

$$f(\mu_u, \mu_d, \mu_s, \mu_e) = \frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e = 0$$
(5.12)

where n_a are the number densities of the specified particles

$$n_a = \frac{\mathrm{d}\Omega(\mu_a)}{\mathrm{d}\mu_a} \tag{5.13}$$

The chemical potentials can be constrained by beta equilibrium. From the reactions $u_d \rightarrow u_u + e^- + ve$ and $u_s \rightarrow u_u + e^- + v\mu$ the chemical potentials must satisfy

$$\mu_d = \mu_s = \mu_u + \mu_e + \nu_{e,\mu} \tag{5.14}$$

As neutrinos can easily escape the quark star, their chemical potential is put to zero.

If we write $\mu_d = \mu_s = \mu$, $\mu_u = x\mu$ and $\mu_e = (1 - x)\mu$, we can plug this into equations 5.12 and 5.13 and solve the resulting equation $f(\mu, x) = 0$ numerically. In the programme the solving algorithm used is the Golden algorithm of Numerical Recipes in C (Press et al., 1996). In a certain range of chemical potentials there are multiple values of x, where x is a parameter that multiplies μ to obtain μ_u (see figure 5.2). Following x as the chemical potential decreases easily identifies the point in the diagram that has to be used, but sometimes the solving algorithm fails in this task. Improving the algorithm probably solves this problem, but simply allowing some errors and removing those was much quicker.

5.3 The equation of state from perturbative quantum chromodynamics

Before we move to solutions of the TOV equation, we first discuss the equation of state of quark matter obtained using perturbative QCD. In this section, we will discuss the equation of state obtained assuming three massless quarks up to second order, and compare it with an equation of state of first order where one massive quark is assumed.

5.3.1 Equation of state of three massless quarks

In figure 5.4 are the pressure-density diagrams obtained from the equation of state assuming three massless quarks to second order perturbation theory. In figure 5.3 are the energy density and pressure of a quark gas relative to an ideal gas plotted against chemical potential. The behaviour of the coupling constant depends on the variable Λ , while the only physical parameter of the system is the chemical



Figure 5.2: Values of the number density equation 5.12, rewrited as described below equation 5.14, as function of x and μ (left pane) and as function of x at a fixed value of $\mu = 375$ MeV (right pane). The solving algorithm should find the rightmost root.



Figure 5.3: *Pressure and energy density calculated from the thermodynamic potential obtained by perturbative QCD to two loop order (equation 4.41).*

potential μ . As discussed in section 3.4, the best we can do is to assume a relation $\Lambda = c\mu$, with c a free parameter. This parameter can then be fitted by other methods.

It is clear that for a finite chemical potential, the pressure vanishes while the energy density remains finite. This behaviour can also be obtained by using a bag model. However, in a bag model the absolute difference between ϵ and P is constant, while in our approach the difference $P - \epsilon$ may, and indeed does, vary. As the variation is not very large, it can be concluded that the MIT bag model is a reasonable model for quark matter. Another reassuring observation is that for infinite chemical potential the equation of state approaches the equation of state of an ideal gas, as seen in figure 5.4. This shows that asymptotic freedom indeed occurs, as can also be concluded from figure 3.4.

Finally, when we plot the interaction strength versus the chemical potential (see figure 3.4) we can see that for $\Lambda \gtrsim 548$ the pressure vanishes before $\alpha < 1$, so the use of perturbative QCD to the surface of the quark star models is allowed for both the model using $\Lambda = 2\mu$ and the model using $\Lambda = 3\mu$. These chemical potentials are used as their pressure-density relation compares to that of the MIT bag model with values close to the accepted bag pressure $B^{1/4} \simeq 150$ MeV. In Fraga et al. (2001)



Figure 5.4: Pressure as function of energy density. Note that the curves with interaction can be quite accurately described by an ideal gas minus a vacuum pressure. This indicates that the MIT bag model is a reasonable approximation for quark matter. See equation 5.15 and below for a discussion.

it is shown that the models compare very well to a non-ideal bag model

$$\Omega(\mu) = \frac{N_f \mu^4}{4\pi} \alpha_{\text{eff}} + B_{\text{eff}}$$
(5.15)

with $B_{\text{eff}}^{1/4} = 199 \text{ MeV}$ and $\alpha_s = 0.628$ for the model using $\Lambda = 2\mu$. The model using $\Lambda = 3\mu$ compares to $B_{\text{eff}}^{1/4} = 140 \text{ MeV}$ and $\alpha_s = 0.626$.

Higher values of c in $\Lambda = c\mu$ represent weaker interactions. This follows from the lower effective bag pressure needed to fit this model to equation 5.15, but can also be seen in figure 5.4. In this figure, the line representing $\Lambda = 3\mu$ lies much closer the the line representing ideal gas. Higher values of $c \gg 3$, the effective bag pressure becomes too low to explain the low energy behaviour of QCD. For $c \ll 2$ the pressure of quark matter remains small for even very high densities, which is also in contrast with calculations.

5.3.2 Equation of state for one massive quark flavour

In figure 5.5 the pressure and energy density as a function of chemical potential are plotted for one massive and two massless quark flavour to first order in α_s . In figure 5.7 the pressure is plotted as a function of energy density. From the plots we see that the strange quark mass has influence only for low chemical potentials or low pressure and energy density. this is expected for two reasons. First, as the chemical increases, so does average momentum of the particles. Mass terms can only be ignored if the momentum $\mathbf{k}^2 \gg m^2$, which is the case only for high chemical potential. The second reason for the massless approximation to become better at high chemical potential is illustrated in figure 4.1. The running mass of the strange quark decreases as a function of chemical potential. As at high chemical potentials is better justified.



Figure 5.5: Pressure and energy density as function of chemical potential for an equation of state assuming one massive and two massless quark flavours (equation 4.42) compared to the equation of state for three massless quarks. Both equations of state are calculated for $\Lambda = 2\mu$, from a thermodynamic potential with corrections up to one loop order.



Figure 5.6: *Energy density as a function of chemical potential, calculated by the analytical calculation (equation 5.21).*



Figure 5.7: Pressure as function of energy density for an equation of state assuming one massive and 2 massless quark flavours, compared to the case of three massless quark flavours. Both equations of state are computed for $\Lambda = 2\mu$, from a thermodynamic potential with first order loop corrections.

These plots differ from the analysis in Fraga and Romatschke (2005), where I think there is an error in the energy density plot. I have checked this using an analytical calculation of the massless case to first order. I compared these with the results both of my calculations and the results in Fraga and Romatschke (2005). The numerical calculations for three massless quark species to one loop order have been executed with the same programme as the numerical calculations for one massive quark, except that I put the scale factor \hat{m} in the running mass (equation 4.43) to a tiny number. I have also checked that this agrees with the result of using the massless first order equation for all three quarks.

The energy density can be calculated analytically. The energy density is given by

$$\epsilon(\mu) = -P(\mu) + \mu n(\mu) \tag{5.16}$$

with

$$P(\mu) = -\Omega(\mu) \tag{5.17}$$

$$n(\mu) = \frac{\partial \Omega(\mu)}{\partial \mu} \tag{5.18}$$

The thermodynamic potential for a noninteracting gas is given by

$$\Omega_f = \frac{\mu^4}{4\pi^2} \tag{5.19}$$

and Ω for one flavour is given to one loop order by

$$\Omega = \frac{\mu^4}{4\pi^2} \left(1 - 2\frac{\alpha_s(\mu)}{\pi} \right) \equiv \Omega_f \left(1 - 2\frac{\alpha_s(\mu)}{\pi} \right)$$
(5.20)



Figure 5.8: *Mass-radius plot obtained by solving the structure equations 2.1 and 2.2 and the equation of state of three massless quarks (equation 4.41) to two-loop running.*

Using this we can write an expression for ϵ/ϵ_f

$$\frac{\epsilon}{\epsilon_f} = 1 + \frac{2}{3\pi} (-3\alpha_s(\mu) - \mu\alpha'_s(\mu))$$
(5.21)

Using $\alpha(\Lambda)$ to one loop order, with a scaling factor $\Lambda = 2\mu$ one obtains an energy density as plotted in figure 5.6. This clearly coincides with the numerical result of figure 5.5(b).

5.4 Numerical solutions to quark star models using a perturbative QCD equation of state

In this section I will present the results of numerical solutions of the TOV equations using an equation of state based on perturbative QCD. As we will see, the model strongly depends on the renormalisation scale Λ . Unfortunately, there are few successful lattice QCD calculations that can be used to fix Λ , as lattice QCD at high density is plagued by large imaginary components of the chemical potential to which there is currently not a good solution (see Ivanov et al., 2005, for a lattice calculation of a QCD inspired model).

5.4.1 Quark star models using an equation of state of three massless quarks

Using the results of section 5.3.1 it is possible to solve the TOV equations (equations 2.1 and 2.2). This can only be done numerically. Using the computer programme described in section 5.2.3 I have obtained the results plotted in figure 5.8. This figure is plotted with a precision on μ and M of 10^{-4} . The radius is a running variable so its precision cannot be specified. However, as shown in figure 5.9, a comparison between the precision I used and a precision 10 times less shows that the error is very small.



Figure 5.9: Comparison of a programme run with a precision of 10^{-3} and 10^{-4} for the model using three massless quarks to two-loop running.

For low mass quark star models there is a power law behaviour $R \propto M^3$, as was expected from the analysis of section 5.1.1. For larger mass the gradient steepens, until a maximum is reached when $dM/dr = \infty$. After this point r decreases as M still increases. The maximum stable model is reached when dM/dr = 0. After this point no stable solutions can be found using the equation of state of the model. A list of turning points can be found in table 5.1. Note the change in central pressure between maximum radius and maximum mass. For quark stars with small masses a slight increase in central pressure makes the star much more massive, while for larger masses a much larger change in central pressure is needed for a similar change in mass. This can be understood from the discussion in section 5.1.1. For low mass models gravity is unimportant and the matter behaves as an incompressible medium. At larger masses, the effect of gravity tends to compress the matter and to let it behave more like degenerate matter under gravity.

As is very clear from this plot, the scale Λ plays a critical role in the maximum mass. Both the maximum mass and maximum radius are $\propto 1.5$ times larger. Also the internal pressure is much lower. This makes sense as a larger scaling of Λ with μ leads to a lower interaction strength.

Point	<i>R</i> (km)	$M(M_{\odot})$	$\rho_c (\text{MeV/fm}^3)$
$R_{\rm max}, \Lambda = 2\mu$	6.18	1.01	4.15×10^{3}
$M_{\rm max}, \Lambda = 2\mu$	5.96	1.09	9.22×10^{3}
$R_{\rm max}, \Lambda = 3\mu$	12.9	2.05	9.02×10^{2}
$M_{\rm max}, \Lambda = 3\mu$	12.4	2.23	2.11×10^{3}

Table 5.1: List of maximum masses and radii for $\Lambda = 2\mu$ and $\Lambda = 3\mu$ using an equation of state of three massless quarks.



Figure 5.10: Mass-radius plot obtained by solving the structure equations 2.1 and 2.2 and the equation of state of one massive quark flavour (equation 4.42) compared to the equation of state of three massless quarks. Both models are calculated up to first order in α_s . The coupling constant is also calculated to one loop running. Both models use a scaling $\Lambda = 2$.

5.4.2 Quark star models using an equation of state of one massive quark flavour

In figure 5.10 the effect of giving one quark flavour a mass is shown. This equation of state is compared to an equation of state for three massless quarks. Both equations of state contain first order loop corrections. Comparing to the results of the previous section, second order contribution leads to ~ 1.5 times smaller masses and radii. Inclusion of a running strange quark mass results in a change in maximum mass and radius with a factor of about 1.5, in effect giving a similar contribution as including the second order correction. Although I use a different equation of state as in Fraga and Romatschke (2005), as is explained in 5.3.2, their conclusion that the mass of the strange quark should not be neglected is still justified by my plots. However, the masses obtained are rather different. Using my equation of state the maximum masses and radii are about half the masses and radii of Fraga and Romatschke (2005).

These models were again calculated with a precision of 10^{-4} on μ and M, which also leads to small errors on R as shown in figure 5.11. In table 5.2 a list of maximum masses and radii obtained from figure 5.10 is given.

Point	<i>R</i> (km)	$M(M_{\odot})$	$\rho_c (\text{MeV/fm}^3)$
$R_{\max}, m_s = 0$	9.13	1.92	2.01×10^{3}
$M_{\rm max}, m_s = 0$	8.96	1.99	3.48×10^{3}
$R_{\max}, m_s \neq 0$	6.59	1.02	4.00×10^{3}
$R_{\max}, m_s \neq 0$	6.27	1.14	1.01×10^{4}

Table 5.2: Maximum mass and radius for a model using an equation of state assuming three massless quarks and model using an equation of state assuming one massive quark flavour, all to one loop running.



Figure 5.11: Comparison of a programme run with a precision of 10^{-3} and 10^{-4} for the model using one massive quark flavour.

When using the programme to calculate quark star models based on an equation of state with one massive quark flavour, some models gave odd results. These results were not on the mass-radius relation as plotted in 5.10. I have deleted these points as they are the result of an error in the numerical calculation. This problem is discussed in section 5.2.3.

5.5 Conclusions on the mass-radius relation

The quark star models described above certainly lead to realistic predictions mass and radius. The radii of quark stars are just below radii of neutron stars with similar masses, while also maximum allowed masses are in the range of observations (see figure 8.1). As discussed more comprehensively in the discussion, chapter 8, there seems no observation based constraint on mass or radius to exclude equations of state based on perturbative QCD as equations of state for compact stars.

Unfortunately, the allowed mass and radius for a quark star is very dependent to the equation of state. The difference in the mass-radius relation for a scaling $\Lambda = 2\mu$ and $\Lambda = 3\mu$ is about a factor 1.5. Also the mass-radius relation based on perturbative QCD up to one loop corrections of three massless quarks is very different from the mass-radius relation from perturbative QCD to two loop corrections. Apparently the two loop correction is still very significant. This leads to the idea that also the three loop corrections might have strong influence on the equation of state and the models based in it. Unfortunately, up to today the three loop correction has not been calculated. As the two loop corrections where calculated now some thirty years ago, apparently the calculations of the three loop corrections are much more complex.

An important observation of the last section is that ignoring the strange quark mass is not justifiable by calculations. Inclusion of the mass leads to very different mass-radius relations, and there is no reason to assume that this situation will be different up to two loop corrections.

Calculations based on phenomological models, such as the MIT Bag model and the NJL model

also give similar mass-radius relations. However, these mass-radius relations are also strongly dependent on the free parameters in the theory. The MIT Bag relation strongly depends on a bag pressure that can only be fitted at low density. Mass-radius relations based on an NJL equation of state depend on the four fermion coupling strength, again a parameter that can only be fitted at low density.

Finally, as shown by Rüster and Rischke (2004), inclusion of colour superconductivity might be a strong influence on the mass-radius relations. For a strong coupling in the NJL model colour superconductivity occurs in the whole region of weakly interacting quark matter, leading to very different mass-radius relations. Their result is in very strong contradiction with Buballa (2005), who finds only a few percent change in the mass-radius relation due to colour superconductivity.

CHAPTER 6

Theory of superconductivity

In this chapter the theory of superconductivity will be reviewed. Superconductivity can arise when an attractive interaction is available that can bind two fermions. Cooper (1956) showed that such an interaction can, for any interaction strength, always lower the ground state. This can be explained by realising that the creation of Cooper pairs lowers the Fermi levels of the system. The chemical potentials of degenerate fermions are about the Fermi energy: $\mu \simeq \epsilon_F$, thereby creating an energy gain. This allows even very weak interactions to create these pairs and thereby superfluid or superconducting behaviour.

In the first section we will see the consequences when the attractive potential is caused by electromagnetic interactions. The collective behaviour will strongly influence the electromagnetic fields in the system, causing superconductivity. We will also explore the effect when there is an attractive potential between neutral particles, causing in superfluidity.

The second section is about colour superconductivity. We will see that the strong force can also create an attractive potential between fermions, thereby allowing the fermions to form Cooper pairs. Due to the more complicated nature of the strong force different types of Cooper pairs will form, creating a multitude of colour superconducting states. We will explore the effects of these states on electromagnetic fields and rotation.

6.1 Superconductivity in neutron matter

The microscopic theory for superconductivity in the J = 0 channel is the BCS theory (Bardeen et al., 1957). This theory applies to low temperature superconductors and probably proton-superconductivity as expected in neutron stars. The theory requires an attractive interaction that allows the formation of Cooper pairs. Pairs are formed by fermions with opposite momentum near the Fermi surface, the total pair momentum is thus zero. This results in an isotropic condensate of Fermi pairs.

In Earth superconductors these pairs are electrons, and the attraction that binds them is a phononexchange interaction. Qualitatively this effect can be understood by realising that at a microscopic level a lattice is not a homogeneous entity. When an electron passes through a lattice, it attracts positive nuclei through the electromagnetic interaction. Due to the much larger mass of these nuclei, they respond much more slowly on the movements of the electron, and are still somewhat displaced when the electron is long gone. This creates a slight positive charge in the wake of a moving electron, allowing it to bind to another electron.

Since this interaction is very weak, binding occurs on very long length scales. In Earth supercon-

Chapter 6: Theory of superconductivity

ductors, these length scales often reach many times the average electron separations distance. This results in a collective behaviour since the wave functions of the pairs will be strongly overlapping. The amount of overlapping can be described by a length scale, the coherence length. The coherence length describes the average size of the wave function of electrons in the superconducting state.

One good example of such collective behaviour is the Meissner effect. This effect describes the tendency of a superconductor to drive out any magnetic field. In low temperature superconductors the coherence is such that there only exist two situations when a magnetic field is applied. For low fields, the superconductor will perform perfect diamagnetism and expel the complete field. At a certain material dependent field strength superconductivity is destroyed and the field can pass the now normal conductor. This behaviour is known as type I superconductivity.

6.1.1 Type I superconductivity

It can be understood by a length scale comparison. The first length scale, the coherence length, is already discussed. Another important length scale is the penetration depth. This length scale give the characteristic distance it takes for the superconductor to lower the field to 1/e of its vacuum value. Usually, this is only a very short distance, stretching a few tens of lattice sites. Using as notation $\lambda(T)$ as the penetration depth and $\xi(T)$ as the coherence length, type I superconductivity arises when (Tinkham, 1996)

$$\kappa = \frac{\lambda(T)}{\xi(T)} < \frac{1}{\sqrt{2}} \tag{6.1}$$

Otherwise, formation of flux tubes is more energetically favourable than completely expelling the magnetic field.

Yet, it is possible for magnetic fields to penetrate type I superconductors by forming nonsuperconducting domains, as shown in figure 6.1



Figure 6.1: When an infinite slab of type I superconducting material blocks an electromagnetic field, it is energetically more favourable to allow the field to pass. For this, formation of domains where $H = H_c$ will form, destroying superconductivity there but allowing the field to pass (image from Tinkham, 1996).

for an infinite two-dimensional slab of superconducting material. These domains differ from the fluxoids present in type II in that they are larger and contain a larger magnetic flux that is not quantised (see section 6.1.2). These domain structures form in non-trivial geometries. In a sphere, for example, the orthogonal field on the poles is of a much higher value than on the equator, where it drops to zero. This can lead to the situation where at the poles the critical field is reached, while at the equator naive analysis suggests that the magnetic field can be expelled. In this situation various domains of a non-superconducting state will form, allowing the field to pass.

To understand this behaviour, we need to study the energetics of this situation. Following the outline of Tinkham (1996), we need to compare the energy of a normal state with magnetic field to a superconducting state with an expelled field. The energy of a sample of volume V in the normal state is

$$F_n = V f_{n0} + V \frac{\mu_0 H^2}{2} + V_{\text{ext}} \mu_0 H^2$$
(6.2)

with f_{n0} the free energy density of the normal state without a field, V_{ext} the volume of the field and H the auxiliary magnetic field. For the superconducting state the free energy is given by

$$F_s = V f_{s0} + V_{\text{ext}} \frac{\mu_0 H^2}{2}$$
(6.3)

with f_{s0} the free energy density of the superconducting state. Combining these yields

$$F_n - F_s = V(f_{n0} - f_{s0}) + V \frac{\mu_0 H^2}{2} = V \frac{\mu_0 H_c^2}{2} + V \frac{\mu_0 H^2}{2}$$
(6.4)

since $f_{n0} - f_{s0} = \mu_0 H_c^2 / 2$.

Now we apply this result to a sphere. Outside the sphere the field satisfies the free Maxwell equations

$$\nabla \cdot \mathbf{B} = \nabla \times \mathbf{B} = \nabla^2 \mathbf{B} = 0 \tag{6.5}$$

where $\mathbf{B} \to H$ for $r \to \infty$.

Inside the sphere we have two possibilities. If $H \ll H_c$ the field will be completely expelled. Yet for stronger field different behaviour can occur. If $H < H_c$ then **B** = 0 inside the superconducting medium. This leads to the extra boundary condition

$$B_{\perp} = 0$$
 at $r = R$

with R the radius of the sphere. The solution to this problem is then

$$\mathbf{B} = \mu_0 \mathbf{H} + \frac{\mu_0 H R^3}{2} \nabla \left(\frac{\cos \theta}{r^2} \right)$$
(6.6)

with θ the polar angle with respect to **H**. From this equation we can obtain the tangential component

$$B_{\parallel} = \frac{3}{2}\mu_0 H \sin\theta \tag{6.7}$$

Now $B_{//} > \mu_0 H$ for 0.73 < θ < 2.41. This means that while the average field can still be below H_c , for parts of the sphere $H > H_c$. As soon as $H = 2/3H_c$ the field at the equator will reach H_c . From that moment, certain regions of the sphere have to be in the normal state. The coexistence of a superconducting state with non-superconducting domains can thus occur over the range of

$$\frac{2}{3}H_c < H < H_c \tag{6.8}$$

As shown in Tinkham (1996) the magnetisation increases linearly from zero to H_c as the applied field increases from $2/3H_c$ to H_c .

6.1.2 Type II superconductivity

When $\kappa > 1/\sqrt{2}$, with κ as in equation 6.1, a very different behaviour occurs. The transition can be derived when realising that there actually exist two critical fields. One is the critical field H_c as discussed in the previous section, the other is a critical field H_{c2} that indicates that stable vortices can penetrate the material.

The second field can be obtained using the linearised Ginzburg-Landau equation. Again, the derivation closely follows Tinkham (1996). This equation reads

$$\left(-i\nabla - \frac{2\pi\mathbf{A}}{\Phi_0}\right)^2 \psi = -2m^* \alpha \psi \equiv \frac{\psi}{\xi(T)}$$
(6.9)

In this equation m^* is the effective mass, α the electromagnetic coupling constant, **A** the magnetic vector potential and Φ_0 a quantised amount of magnetic flux, as will be shown in equation 6.20. The last equivalence is in the definition of the characteristic length $\xi(T)$

$$\xi(T)^2 = \frac{1}{2m^*|\alpha(T)|} \tag{6.10}$$

while ψ is the Ginzburg-Landau (GL hereafter) complex order parameter, describing the fraction of the material in the superconducting state.

Using this framework, we can derive when nucleation of superconductivity arises in a bulk sample. The field is taken parallel to the z axis, with a possible gauge choice

$$A_v = Hx$$

with all other components of **A** equal to zero. We furthermore choose $\epsilon_0 = \mu_0^{-1} = 4\pi$. Then we can expand the left hand side of equation 6.9 to obtain

$$\left[-\nabla^2 + \frac{(4\pi)i}{\Phi_0}Hx\frac{\partial}{\partial y} + 2\pi^2 \left(\frac{H}{\Phi_0}^2\right)x^2\right]\psi = \frac{1}{\xi^2}\psi$$
(6.11)

with $\Phi_0 = 1/2e$. This solution only depends on *x*, so we make an Ansatz of the form

$$\psi = e^{ik_y y} e^{ik_z z} f(x) \tag{6.12}$$

which we insert into equation 6.11 to obtain

$$-f''(x) + 2\mu_0 \pi^2 \left(\frac{H}{\Phi_0}\right)^2 (x - x_0)^2 f = \left(\frac{1}{\xi^2} - k_z^2\right) f$$
(6.13)

with

$$x_0 = \frac{k_y \Phi_0}{(4\pi)H}$$
(6.14)

When we multiply equation 6.13 with $1/2m^*$, we get the Schrödinger equation for a particle of effective mass m^* in a harmonic oscillator with force constant

$$\left(\frac{2\pi^2}{m^*}\right) \left(\frac{H}{\Phi_0}\right)^2$$

This equation can be solved an gives energies

$$\epsilon_n = \left(n + \frac{1}{2}\right) \left(\frac{2eH}{m^*c}\right) \equiv \left(n + \frac{1}{2}\right) \omega_c \tag{6.15}$$

These solutions have to be equated with the right hand side of equation 6.13. From this, the field H is given by

$$H = \sqrt{\frac{1}{4\pi^2}} \frac{\Phi_0}{2n+1} \left(\frac{1}{\xi^2} - k_z^2 \right)$$
(6.16)

The maximum of this function is found for n = 0 and $k_z = 0$, then equation 6.16 yields

$$H_{c2} = \frac{\Phi_0}{2\pi\xi_0} = \frac{4\pi\lambda^2 H_c^2}{\phi_0} = \sqrt{2}\kappa H_c$$
(6.17)
From equation 6.17 it can be seen that for $\kappa > 1/\sqrt{2}$ the field $H_{c2} > H_c$. This leads to the possibility of nucleating superconductivity before it is energetically favourable to drive out the entire magnetic field. In the step from equation 6.16 to equation 6.17 we have used the definitions

$$\xi(T) = \frac{\Phi_0}{2\sqrt{2}H_c(T)\lambda_{\text{eff}}(T)}$$
(6.18)

$$\kappa = \frac{\lambda_{\text{eff}}(T)}{\xi(T)} = \frac{2\sqrt{2\pi H_c(T)\lambda_{\text{eff}}^2(T)}}{\Phi_0}$$
(6.19)

For fields $H : H_c < H < H_{c2}$ it is possible to have a coexisting magnetic field and superconductor. The field penetrates the superconductor in fluxoids. A peculiarity is that these fluxoids can contain only a very specific amount of field

$$\Phi' = n \frac{2\pi}{e^*} = n \Phi_0 \tag{6.20}$$

Here e^* is the effective charge of the pair, which is ideally 2e for a pair of two particles with charge e. As it is energetically favourable to form as many fluxoids as possible, n will often be 1.

The quantisation of flux can be understood from the Ginzburg-Landau idea of having a singe complex GL order parameter ψ . In order for ψ to be single-valued, only a phase change for circling the fluxoid of $2\pi n$ is allowed.

From this it can also be seen that these fluxoids are topological defects of the smooth superconductor. It is not possible to bring such a defect into being by a continuous gauge transformation of the fields. The only way of inserting a fluxoid is by creating one at the edge of the superconductor.

6.2 Colour superconductivity

When a material becomes superconducting to electromagnetic fields, the U(1) symmetry of electromagnetism is globally broken due to electron pairing. This results in the photons acquiring an effective mass, damping the electromagnetic field at very short length scales. At the same time the pairing of electrons in Cooper pairs puts them in a bosonic ground state, generating a sort of electronic Bose-Einstein condensate in the material. The result is that any attempt to apply an electromagnetic field will be cancelled by the system, in effect allowing an electric current to flow without any resistance whilst expelling all magnetic fields. For this mechanism to happen an attractive channel for the pairing particles is needed. In low-temperature superconductors phonon exchange is providing the required mechanism that creates an attractive potential for electrons in the superconductor.

A similar mechanism can also happen to fermions carrying a colour charge. The $3 \otimes 3$ representation of the SU(3) symmetry group of quark interaction can be decomposed in $6 \oplus \overline{3}$, where the $\overline{3}$ channel is attractive. This can be seen by realising that single gluon exchange is attractive between two quarks antisymmetric on colour. This is the case for the $\overline{3}$ colour channel (Rajagopal and Wilczek, 2000). That strong binding occurs in this channel can be understood by realising that for strong pairing to happen the wave function must be symmetric in *r*. As $\overline{3}$ is antisymmetric in colour for states in this representation a wave function can be symmetric in position space, leading to a bound state. For stronger coupling constants the there can also be a contribution of the instanton interaction, which is also attractive in this channel.

Since a quark condensate typically consists of more than one particle species, and the charge is a root from the 3 representation of the SU(3) symmetry group, it shows a more complex behaviour than electromagnetic superconductivity. There are quite a few distinct phases present, which mainly differ in the particles participating in the pairing.

6.2.1 Colour-Flavour Locked phase

At very high densities well above the strange mass, weak equilibrium will guarantee that all three low mass quark flavours will be present. In that case, the lowest energy attainable is achieved by pairing all three quarks in the so called colour-flavour locked phase. This condensate is of the following form (Rajagopal and Wilczek, 2000)

$$\left\langle \psi_{iL}^{a\alpha}\psi_{jL}^{b\beta}\epsilon_{ab}\right\rangle = -\left\langle \psi_{iR}^{a\alpha}\psi_{jR}^{b\beta}\epsilon_{ab}\right\rangle = \Delta(p^2)\epsilon^{\alpha\beta A}\epsilon_{abA}$$
(6.21)

Here $\{a, b\}$, $\{\alpha, \beta\}$ and $\{i, j\}$ are indices for respectively spin, colour and flavour. In the last term the Levi-Civita tensor clearly links colour and flavour, hence the name colour-flavour locked. The Levi-Civita tensor is an antisymmetric tensor, making the states $\langle \psi_{iL}^{a\alpha} \psi_{jL}^{b\beta} \epsilon_{ab} \rangle$ antisymmetric in both colour and flavour indices. The binding energy of these pairs is very high, with expected superconducting gaps of the order of 50 – 100 MeV.

Consequence of this locking is a symmetry breaking similar to that in the BCS theory of superconductivity. There is a non-zero expectation value for condensate pairs, which break the symmetry of massless QCD to a much smaller group (Rajagopal and Wilczek, 2000)

$$SU(3)_{colour} + SU(3)_L + SU(3)_R + U(1)_B \rightarrow SU(3)_{colour+L+R} \times Z_2$$
(6.22)

An interesting question in this context is the effect of symmetry breaking on the symmetry group of electromagnetism. Electromagnetism couples to the different quarks with different field strengths $(\frac{2}{3}e$ for u and $-\frac{1}{3}e$ for d and s), so the general U(1) symmetry of electromagnetism is definitely broken by quark cooper pairing. However, like in the analogous theory of electro-weak symmetry breaking, a linear combination respecting U(1) symmetry survives. The symmetry is (Alford et al., 2000)

$$\widetilde{Q} = Q + \eta T_8 \tag{6.23}$$

with $\eta = 1/\sqrt{3}$. Here, \widetilde{Q} is the charge in the surviving symmetry and Q and T_8 are

$$Q = e \begin{pmatrix} \frac{2}{3} & 0 & 0\\ 0 & -\frac{1}{3} & 0\\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \qquad T_8 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2 \end{pmatrix}$$

By this construction, all Cooper pairs carry no charge in this new symmetry. As the condensate is \overline{Q} neutral, this new symmetry will be unbroken. Part of the electromagnetic field will be expelled by the Meissner effect, but a large fraction of the field can pass as a \widetilde{Q} field. For a CFL-phase with no electron contribution this would lead to the diffraction of light waves similar to glass or diamond.

6.2.2 Two-quark superconducting phase

At lower energies the comparatively large mass of the strange quark will spoil the symmetries allowing CFL-superconductivity. As weak equilibrium does not guarantee equal number density of u, d and s in this case, the flavour symmetry is not possible. Still, given their small mass, assuming weak equilibrium between u and d is reasonable. In this case there exists the possibility of the u and d quarks pairing in a two species superconducting phase, or the 2SC-phase.

In the 2SC phase, pairing of flavour singlets (ud - du) occurs. This pairing breaks the full SU(3) symmetry of QCD down to a SU(2) subgroup, leaving one colour unpaired. Although in the case of massless quarks the CFL-phase is energetically favoured, for massive quarks the 2SC-phase can be the

ground state at certain energies. The strong interaction of quarks leads to a large gap $\sim 50 - 100 \text{ MeV}$ also for the 2SC-phase (Alford et al., 2000, Buballa, 2005)

There are a few important distinctions between the CFL phase and the 2SC phase. This phase leaves the global flavour symmetry $SU(2)_L \otimes SU(2)_R$ intact, and also the global U(1) symmetry of rotation survives. Therefore, the 2SC-phase is not a superfluid. Again, the U(1) symmetry is a linear combination of the photon with one of the eight gluons. Furthermore, this phase respects chiral symmetry. Since this symmetry is broken at low densities, there has to be a phase transition from low densities to the 2SC phase. Increasing the density further, chiral symmetry will be broken again by the CFL phase, requiring another phase transition. In Rajagopal and Wilczek (2000) it is argued that this phase transition has to be first order, as the order parameter of the $\langle us \rangle$ and $\langle sd \rangle$ to be nonzero requires gaps Δ_{us} and Δ_{ds} greater than $m_s^2/2\mu$.

In that same article it is calculated that at $\mu = 400 - 500$ MeV, an energy scale relevant for quark stars, both the CFL phase and 2SC phase are possible, depending on the interaction strength. The strength of the interaction is currently not known precise enough to predict the favoured state.

6.2.3 One flavour pairing

It is also possible that one flavour will pair with itself. In the 2SC phase, for example, the strange quark is left unpaired from the ud - du condensate. However, this does not forbid the existence of a $\langle ss \rangle$ condensate. In this case, the pair needs an angular momentum J = 1, since pairing if similar particles with angular momentum J = 0 would violate the exclusion principle. This results in a pairing with a small gap compared to the s-wave colour superconducting states like CFL and 2SC, estimated to be of the order of a few hundred keV.

6.2.4 Other colour superconducting phases and the quantum chromodynamics phase diagram

The analysis of the CFL phase assumes a symmetry between the three participating quarks. However, the strange quark does possess a considerable mass, possibly breaking the symmetry of CFL. At lower energies, the mass of the strange quark might prevent pairing of two quarks with opposite momenta due to unequal Fermi momenta. One of the possibilities is the existence of a gapless phase. However, studies (Alford and Rajagopal, 2006, Rajagopal and Sharma, 2006, and references therein) has shown that the gapless phase does not represent the state of lowest free energy, and that the ground state must be formed by a pairing of particles into Cooper pairs with a net momentum. This phase was first studied by Larkin and Ovchinnikov (1964) and Fulde and Ferrell (1964) and is known as the LOFF-phase. This results in breaking translational invariance, hence the name crystalline colour superconductivity.

Another possibility is the formation of a Kaon condensate of K^0 particles formed by binding two Cooper pairs into a particle like excitation with the quantum numbers of a Kaon (see Alford and Rajagopal, 2006, and references therein). According to this reference the Kaon condensate is easily destroyed by instanton effect. This conclusion is argued by Warringa (2006) who performed a detailed calculation of the instanton effect on the Kaon condensate and does find a stable CFL-K⁰ phase at low temperature.

Using the NJL effective model it is possible to calculate the QCD phase diagram (figure 6.2). In the NJL effective theory the one-gluon exchange interaction is replaced by an effective four-fermion interaction. This simplification allows computation of various parameters, like the order parameter of the superconducting phases. The strength of the four-fermion interaction is not well determined, leading to a significant uncertainty in this diagram. However, the general structure of figure 6.2 does not depend on this.

The existence of the different phases can also be understood from a physical perspective. At zero temperature and density quarks are confined and chiral symmetry is broken. At very high temperatures both these symmetries are restored, making phase transition necessary. At very high densities the CFL-phase is expected to be the ground state of matter. This condensate also possesses a broken chiral symmetry, since colour and flavours are linked. Yet, confinement is lifted, and thereby the chiral symmetry breaking mechanism of low density is restored. This leads to the possibility of a phase transformation from confined matter to a chiral symmetric quark condensate.



Figure 6.2: The QCD phase diagram. At high temperatures formation of a quark-gluon plasma occurs. At high densities neutral quark matter in one of the colour superconducting states is expected (image from Formal, 2007)

CHAPTER 7

Magnetic fields in hybrid stars and quark stars

In the previous chapter we have discussed the properties of electromagnetically and colour superconducting matter. Now, we will explore the consequences for the magnetic properties of compact stars containing quark matter. As is found by Alford et al. (2000) deconfined quark matter responds very differently to electromagnetic fields than normal matter. In more recent work (Ferrer and de la Incera, 2006a,b) it is even suggested that colour superconducting quark matter may act as a paramagnet. This in sharp contrast to the perfect diamagnetism present in electromagnetic superconductors.

7.1 Neutron star magnetic behaviour

Superconductivity has a strong influence on the magnetic properties of neutron matter. One of the main characteristics of a neutron star is its stable field. It is generally accepted that magnetic fields of a pulsar do not decay during its lifetime. Superconducting matter allows field lines to decay only via moving its flux tube to the surface, where the magnetic field line can reconnect and release its energy. This creates a very stable field, especially when combined with the electronic component. This component is not in a superconducting sate but has very good conducting properties, generating a strong Lorentz force on any moving field line (see section 1.2.3 for a discussion).

7.1.1 Fluxoid pinning

For a long time, the assumption has been made that the superconducting medium of a neutron star is of type II. For fields $H : H_c < H < H_{c2}$ it is then energetically favourable to allow a field to exist in a superconductor by forming fluxoids. As these fluxoids are tiny regions of non-superconductivity and thus at that region the ground state energy is lifted by the superconducting gap. To minimise the energy loss, fluxoids will pin to sites where this gap has a minimal value. A superconductor is calculated assuming a smooth background, the defects therein are the pinning sites. In order to be able to pin to such a defect, it has to be larger than the correlation length ξ . For proton matter ξ is very small (Anderson et al., 1982)

$$\xi = \frac{v_F}{\Delta} \tag{7.1}$$

where v_F is the Fermi velocity and Δ is the superconducting gap. Using this relation and equation 6.1 leads to the following condition on the penetration depth of proton matter not to be a type II

superconductor:

$$\lambda(T) < \frac{1}{\sqrt{2}} \frac{v_F}{\Delta} \sim 1 \,\mathrm{fm} \tag{7.2}$$

For an assumed $v_F \sim 100 \text{ MeV}$ and $\Delta \sim 0.5 \text{ MeV}$. The magnitude of λ is typically of the order of the London penetration depth

$$\frac{1}{\lambda^2} = \frac{4\pi e^2 n_s}{m^*} \tag{7.3}$$

with n_s the superconductor carrier density and m^* the effective mass of the superconductor carrier. Assumed values for λ are of the order of 80 fm, in which case type II superconductivity is favoured. According to Buckley et al. (2004) the self-interaction changes this picture and makes type I superconductivity favoured. In Alford et al. (2005), Jones (2006) it is shown that the suggested interparticle interaction is too weak to for type I superconductivity.

Apart from the superconducting proton component a neutron star is also thought to possess a superfluid neutron liquid. A characteristic property of superfluidity is that rotation is accommodated by the formation of vortices containing all angular momentum. The behaviour of a superfluid to rotation is very similar to that of a superconductor to electromagnetic fields. This is not very strange, as the symmetry of two dimensional rotation is also a U(1) symmetry, just as the symmetry of electromagnetism is a U(1) symmetry. Strong evidence for the presence of a superfluid containing vortices is based on glitches, which are thought to occur as a result of vortex pinning in the lattice near the surface. See section 1.1.1 for a discussion about glitches or section 7.2.1 for a detailed discussion of the theory of glitches assuming vortex pinning.

It is possible to calculate the fluxoid and vortex density inside a neutron star. Superfluid vortices are quantised in angular momentum, and line up parallel to the rotation axis. The angular momentum per vortex is

$$\oint_C \mathbf{v} \times \mathbf{d} \mathbf{l} = \frac{2\pi}{m} n \tag{7.4}$$

were n = 1 is the state of lowest energy. Using this one can derive a vortex density (Hsu, 1999)

$$n_{SV} = 10^4 / P(\text{sec}) \text{ cm}^{-2}$$
 (7.5)

leading to 10^4 vortices per cm⁻² for most pulsars and 10^8 per cm⁻² for the fastest spinning millisecond pulsar.

Using the relation for the flux quantum (equation 6.20) and realising that flux is quantised per proton Cooper pair of charge +2e we can also derive a flux tube density

$$n_{FV} = 10^7 B \,\mathrm{cm}^{-2} \tag{7.6}$$

Assuming typical magnetic fields $\sim 10^8$ T this leads to a much higher density of flux tubes than rotational vortices. The flux tube lattice is expected to be much less regular than that of the vortices (Link, 2003, 2006)

Realising that neutrons have an associated magnetic moment and recalling that flux tubes are often created on defects, it is easy to imagine that flux tubes are strongly coupled to these vortices. In Link (2006) it is calculated that the associated energy per vortex-flux tube pairing of ± 5 MeV, depending on whether the vortex and tube are parallel or anti-parallel. This is a strong pinning, making it nearly impossible for the vortices and fluxoids to move independently from each other, whether this is regular motion or slow vortex creep.

Flux pinning consequences

This pinning has two consequences. First, it relates the magnetic field and rotation period. As the star spins down, it needs less and less vortices to contain angular momentum. These vortices can only disappear through the surface, taking the pinned fluxoids with them. This effect is known for a long time and leads to a weakly time dependent magnetic field (Bhattacharya and Srinivasan, 1995)

$$B(t) \propto N_{\nu} = 2 \times 10^{16} P^{-1} \tag{7.7}$$

with *P* the period in seconds and N_v the number of vortices, to which the magnetic field is proportional if the fluxoids are perfectly pinned to these vortices.

For a solitary pulsar spinning down this would lead to a magnetic field decay of (Bhattacharya and Srinivasan, 1995)

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$$B(t) \propto \left(\frac{t}{\tau}\right)^{-1/4} \tag{7.8}$$

with τ the characteristic spin-down timescale, typically $10^6 - 10^7$ years. This slow change is not ruled out by observations. The effect only becomes appreciable when mass transfer and tidal forces can seriously influence the spin. In low mass X-ray binaries a neutron stars with spins of more than 1000 s have been found. This would reduce the magnetic field by a factor of 10^4 or more, and can be used to explain the low magnetic fields of millisecond pulsars thought to originate from these objects. Yet, this result is contradicted by the finding that these slowly rotating neutron stars still have a strong field ~ 10^8 T (Coburn et al., 2006, more discussion in section 1.1.1)

Another effect of this pinning is the strong coupling of the neutron superfluid to the proton superconductor. This proton superconductor is via the magnetic field coupled strongly to the surface. This in effect makes the neutron star a rigid rotator. This has severe implications on higher order perturbations of the rotation. In Link (2003, 2006) it is calculated that in this case the frequency of precession is of the same order as the rotation period, or faster.

For a rigid biaxial body of oblateness ϵ the precession frequency $\bar{\omega}$ in units of ω is $\bar{\omega} = \epsilon \omega$. Models of neutron stars indicate an oblateness $\epsilon \sim 10^6$, which is supported by observation of Stairs et al. (2000). If, however, the neutron liquid would couple perfectly effective to the body via vortex pinning, the precession frequency would change to

$$\bar{\omega} = \epsilon \omega + \frac{L_p}{I_b \omega} \tag{7.9}$$

In this equation L_p is the angular momentum of the pinned superfluid and I_b the moment of inertia of the rest of the object. Since $L_p \simeq I_p \omega$ and therefore $L_p/I_b \omega \simeq I_p/I_b \gg \epsilon$, this will be the dominant contribution to precession frequency. Using this relation, and realising that most of the inertia will be contained in the neutron superfluid, estimates for the precession frequency yield $\bar{\omega} = I_p/I_b \approx 10$, much higher than observations. This result was first obtained in Link (2003). In a later article (Link, 2006) it is shown that also imperfect pinning, known as vortex creep, is inadequate to explain the low precession periods observed. From these articles it seems that a neutron superfluid and a type II superconductor cannot coexist. Instead a type I superconductor is proposed. I will first discuss the possibility of a colour superconducting compact star core, coming back to the possibility of type I superconductivity in section 7.2.

7.1.2 Quark matter influence on magnetic fields

Colour superconductivity and electromagnetism

A possible solution for the precession paradox discussed above is the presence of quark matter in the interior of a compact star. As mentioned in section 6.2, breaking colour symmetry only shields the electromagnetic field marginally. More important, in the CFL phase the presence of an electromagnetic field does not lead to the formation of fluxoids. Rotation however, does lead to vortex formation in the CFL phase.

In the 2SC phase fluxoids might be present (Iida and Baym, 2002), but the density will be much lower as the field is reduced by only ¹/40 (Alford et al., 2000). When rotating matter in the 2SC phase something more interesting occurs, as rotation of this medium can create a London magnetic field (Iida and Baym, 2002)

$$|\mathbf{B^8}| \sim 0.015 \left(\frac{\sqrt{3}}{g_8}\right) \left(\frac{1 \text{ s}}{P_{\text{rot}}}\right) \left(\frac{\mu/3}{300 \text{ MeV}}\right) \text{ T}$$
(7.10)

which is negligible with the surface magnetic field of a neutron star. While an interesting effect it is probably safe to neglect it.

Stability of magnetic fields in quark matter

While quark matter is not an electromagnetic superconductor, in Alford et al. (2000) it is found that it is an extremely good conductor. Ohmic decay times of the magnetic field are in excess of the Hubble time, and therefore easily in range of observations.

A drawback of this view is that a compact star containing quark matter does not have a natural decay of the magnetic field in a binary, as happens with a compact star containing a type II proton superconductor and a neutron superfluid (equation 7.7).

Precession of neutron stars with quark matter

Using the results from equation 7.9, but allowing for a quark matter component, we can calculate precession rates for compact stars containing quark matter. We will use the assumption that quark matter and neutron matter are neatly separated into different regions. This allows us to approximate the relative size compared to the star of the volume were pinning occurs. This assumption is not unreasonable, as it is thought that there might be a sharp transition between hadronic matter and quark matter. Furthermore, I will assume a uniform density, something that is certainly not the case but it does give an upper bound as a useful first approximation.

Suppose our star consists of a quark core up to radius a, a proton superconductor mixed with a neutron superfluid to radius b and an crystalline iron crust to radius c, as in figure 7.1. The surface is accounted for in the oblateness parameter, and is not needed specifically in our calculation. The volumes of the quark core V_q and hadronic mantle V_h are

$$V_q = \frac{4}{3}\pi a^3$$
(7.11)

$$V_h = \frac{4}{3}\pi(b^3 - a^3) \tag{7.12}$$

so assuming uniform density the fraction of the total hadronic mass to total quark mass will be

$$\frac{M_h}{M_q} = \frac{b^3}{a^3} - 1 \tag{7.13}$$



Figure 7.1: Three-layer model of a compact star. Up to radius a a fluxoid-free region is assumed. Between radius a and b fluxoids are present, the iron crust lattice ranges from radius b to c.

If we assume that only the hadronic component couples to the magnetic field via fluxoid pinning, the precession equation 7.9 changes to

$$\bar{\omega} = \epsilon \omega + \frac{L_p}{I_b \omega} \left(\frac{b^3}{a^3} - 1 \right) \tag{7.14}$$

thus only the angular momentum contained in the hadronic mantle contributes to the precession. Using this equation and observed precession rates we can derive a maximum mass fraction of a layer of superfluid and type II superconducting matter. Assume that the weight of the crust is negligible, and that $\bar{\omega} \sim 10^{-6} \omega$. Then the mass fraction of the superfluid and type II superconducting component can be no more than 10^{-6} . Allowing for vortex creep increases this number but will not allow a significant fraction of the star to be in a type II superconducting state (Link, 2006).

This tiny fraction suggests the conclusion that, if type I superconductivity does not occur, quark matter has to exist up to the region where protons and neutrons start forming a lattice, and that nowhere a neutron superfluid fluid can coexists with a type II proton superconductor. This would mean that quark matter is stable from chemical potentials above the chemical potential of the crust lattice.

This all leads to the conclusion that free precession excludes the possibility of type II superconductivity. Type I superconductivity is still possible, with a damping period of about 100 years (Link, 2006). The presence of quark matter also allows precession. As there is very little interaction between a quark matter core and a hadronic crust, much longer damping periods can be expected in this case.

7.2 Consequences of quark matter for pulsar timing observations

7.2.1 Glitches

Due to electromagnetic braking, the rotational frequency of a pulsar decreases in time. The energy and angular momentum are carried away by the electromagnetic radiation. This mechanism is discussed in more detail in section 1.1.1. Also discussed in that section is that sometimes there is a sudden increase in rotational frequency, a glitch. Glitches are most commonly observed in young pulsars and are thought to originate from an angular momentum transfer between core and crust. The theory is that as the star spins down, the superfluid core needs to spin down as well, due to the interaction between the proton superconductor and the neutron superfluid. For a superfluid to spin down, vortices have to be destroyed. This can only happen at the surface, so the vortices move outwards as the star spins down.

However, in the boundary layer between core and crust there is a coexistence of a crust lattice and superfluid neutrons. The vortices tend to pin to lattice sites, as the interactions between neutrons are very strong and the correlation length is very small. This pinning prevents the vortices from moving outwards so a buildup of angular momentum occurs.

A glitch happens as a result of a massive unpinning. The released angular momentum of all the destroyed vortices is transferred to the crust, which angular momentum increases. As only the crust can be observed from Earth it looks like the compact star suddenly spins up.

Since the above described mechanism for glitches has nothing to do with superconductivity, it is still valid when one assumes a type I superconducting proton component. However, as was showed in Shaham (1977) the pinning of vortices to the crust also leads to a strong damping on precession. A proposed solution is that in a freely precessing compact star these vortices are all unpinned. This would mean that glitches do not happen in precessing compact stars.

It is thought that quark matter cannot coexist with normal matter, so for quark matter a different explanation for glitches is needed. Again, we need a mechanism to store and then suddenly release angular momentum. A mechanism to store angular momentum is the vortex creation, which also happens in a quark superfluid. It is suggested in Alford and Rajagopal (2006), Jaikumar et al. (2006a) that a coexistence of one of the many different colour superconducting states makes it possible to internally store angular momentum. Especially the crystalline colour-flavour locked phase (see section 6.2.4) could play an important role. For pure quark stars it is imaginable that such a mechanism would allow an up-spinning crust. Yet, for stars consisting of a normal crust on top of a quark matter core it is not clear how the angular momentum transfer to the crust would occur. In this light it is probably safe to assume that any hybrid star needs at least a small portion of superfluid neutron matter.

7.2.2 Precession

In an article by Stairs et al. (2000) strong evidence for free precession of a compact star is found. As shown in figure 7.2(a) there are timing residuals after performing all timing corrections. In order to test this assumption it is useful to search for any periodicity in the timing residuals. The Fourier transform of the timing residuals can be found in figure 7.2(b). In the article different mechanisms for the observed harmonics are discussed and rejected. This observation is therefore a strong indication of the presence of free precession in this pulsar.

The results indicate a precession period ~ 500 days and an amplitude $\alpha \simeq 3^{\circ}$ (Link, 2003). Using equation 7.9 without the damping term indicates an oblateness $\epsilon \sim 10^{-6}$, a perfectly normal value for a compact star.



(a) Timing residuals from PSR B1828-11



(b) Fourier transform of the timing residuals

Figure 7.2: *Timing residuals and its Fourier transform from PSR B1828-11. Images from Stairs et al.* (2000)

There is some evidence for other neutron stars showing a similar type of precession (Deshpande and Radhakrishnan, 2007, Shabanova et al., 2001). In Stairs et al. (2000) it is also suggested that timing residuals in anomalous X-ray pulsars can be explained by free precession.

Free precession of a compact star occurs due to a non-spherically symmetric distortion. Most probable such a distortion would be an explosion on the surface of the neutron star. A decaying magnetic field can cause major outbursts, and would be a good candidate. If this is the case, precession in the more violent compact stars such as anomalous X-ray pulsars should be more common. If it is possible to make an estimate of how often a distortion occurs, and how many neutron stars undergo free precession, it is possible to derive a damping time. This damping time would tell a lot about the equation of state of the compact star interior as discussed in section 7.1.2.

In the previous section was argued that glitches should not happen in freely precessing compact stars, as for free precession to happen the neutron superfluid has to be unpinned from the crust lattice. Detailed pulsar timing on precessing neutron stars could validate the assumption that precessing neutron stars cannot show glitches. This would be a strong suggestion that the precession model is correct.

CHAPTER 8

Conclusions and discussion

In the introductory chapters we have discussed many different observables of compact stars. There we found several mechanisms to study the equation of state of matter in a compact star. The most direct method is provided by a mass-radius determination. Furthermore, the interaction of matter and magnetic fields dictates the behaviour of the field and thereby influences the star its rotation. Study of precession, spindown and glitches are particular tools to access the star its magnetic properties.

In this thesis the mass-radius relation and the influence of the magnetic field were studied. The results will be discussed in the next sections. Another way of probing the interior of a compact star is by its cooling behaviour. I haven't discussed this behaviour, especially as results obtained by this method are highly degenerate. However, in order to study the interior of compact stars properly, a combination of all mechanisms leaves the fewest degeneracy between different models. For an observational study of these objects, a combination of all these observables is therefore preferred.

8.1 Mass-radius relations

There is still much uncertainty in the quark matter equation of state. This allows for mass-radius relations differing by as much as a factor 1.5 for reasonable values for the renormalisation scale dependence a in $\Lambda = a\mu$. Further study on QCD parameters, both theoretically and experimentally, must be done in order to make better predictions. Especially the calculations of the quark matter thermodynamic potential to three loop order should put more stringent constraints on the mass-radius relation. Furthermore, detailed study of quark matter properties in astrophysical objects may provide measurements that can help fix these parameters.

Comparing the curves with scaling $\Lambda = a\mu$, a = 2, 3 to curves for neutron matter shows that compact stars based made from quark matter are always smaller than their neutron matter counterparts. This has to be, as quark matter is created when neutron matter is compressed to such densities that confinement is no longer present.

The mass-radius relation compares very well to expectations based on the MIT Bag model as discussed in chapter 5.1.1. The quark matter equation of state behaves as a nearly incompressible medium at low densities, at higher densities it starts behaving more like ordinary degenerate matter. Furthermore, the equation of state based on perturbative calculations (figure 5.4) only differs slightly from non-ideal Bag models with a properly fit Bag constant.

Including one massive quark flavour, I have confirmed the result of Fraga and Romatschke (2005) that neglecting the strange quark mass is not a minor perturbation to the final mass-radius relations.

However, I have found a different equation of state for first order perturbation theory and have also found different mass-radius relations. An analytic calculation supports my results.

Depending on the scale dependence expressed through *a*, quark matter is absolutely stable when there is an energy gain for conversion from baryons to quark matter. For quark matter the pressure vanishes at some chemical potential μ_{crit} . For a scaling $\Lambda = 3\mu$, for example, the pressure vanishes at ~ 300 MeV. To compare it to a baryon chemical potential we have to look at the combined potential of three quarks, 900 MeV, and compare this to the baryon chemical potential of iron, 940 MeV (Fraga et al., 2001). In this case, quark matter will thus be absolutely stable. However, as discussed in chapter 2.2, even such stars will have a baryonic crust as the strange quark mass does not allow for neutral quark matter. This charge must be compensated by electrons. At the surface, the the quark matter density drops much faster to zero than the electron density can follow. This creates a considerable electromagnetic potential at the surface of the quark star, able to support a baryonic crust.

For lower values of *a* quark matter is not absolutely stable and hybrid stars are expected. In order to calculate such stars, an equilibrium in pressure and chemical potential between baryonic matter and quark matter must be required in the model. Although this is not computationally difficult, it is an extra source of uncertainty to the models.

The mass limit of quark stars seems not to be any more stringent for quark stars as for neutron stars. The causality limit, stating that the speed of sound should be less than the speed of light, puts boundaries on the equation of state. The maximally hard equation of state is $\epsilon = P$, leading to maximum masses ~ $3 M_{\odot}$. For compact stars this limit is more stringent than the limit based on general relativity, stating that the Schwarzschild radius of an object should be smaller than the radius of object

$$R \gtrsim 3 \frac{M}{M_{\odot}} \,\mathrm{km}$$
 (8.1)

Otherwise, the object would be a black hole.

For our equation of state with scaling $\Lambda = 3\mu$ masses slightly above $2 M_{\odot}$ are possible, close to the limiting mass based on causality. Including the strange quark results in a lower mass, but due to the uncertainty in scaling the maximum mass of quark stars can still be close to the maximum mass of a compact star. On the other hand, based on low energy information on the scaling constant, a somewhat lower maximum mass is more likely.

For hybrid stars the limiting mass is similar to that of a neutron star. Although the above argument may limit the mass of the quark core, it does not pose limits on the baryonic content. It does mean that, if true that the maximum mass for the quark content is smaller than the maximum mass of a compact star, the most massive compact stars must be made of at least a portion of baryonic matter.

Comparing the models with the observed masses of compact stars in binary systems as shown in figure 8.1, none of the compact stars seem to be excluded based on equations of state discussed in this thesis. In all of the objects in this figure there is strong evidence of the presence of a surface, indicating compact stars and not black holes. It is puzzling why most of the compact stars in this figure seem to cluster so close to the Chandrasekhar mass of 1.4 M_{\odot} . One explanation might be that the maximum mass for compact stars is not much higher than the Chandrasekhar mass, allowing only a small mass window in which these objects can reside. However, this result seems to be contradicted by the the discovery of a compact star with a mass $\sim 2.1 \pm 0.2 \text{ M}_{\odot}$ (Nice et al., 2005). Another explanation may lie in the formation history of binary systems, which may somehow favour the formation of light compact stars. Yet, this scenario is in contrast with some of the proposed black hole systems, such as Cyg X-1, which is estimated to have a mass of about 10 M_{\odot} (Abubekerov et al., 2004).



Figure 8.1: Determination of masses of compact stars in binary systems. It seems that masses close to the minimum compact star mass $M \simeq 1.4 M_{\odot}$ are favoured (image from Thorsett and Chakrabarty, 1999)

8.2 Quark matter and superconductivity

The presence of quark matter has important consequences for the behaviour of electromagnetism in compact stars. While neutron matter is probably superconducting with respect to electromagnetism, quark matter is not.

The stability of the magnetic fields is guaranteed for both types of matter. Superconductivity of both type I and type II confine the magnetic field to domains. For field-lines to reconnect these domains have to move through the surface, which is very difficult as the Lorentz force of the free electrons in the compact object drive the domains back. In quark matter the extreme conductivity of free electrons guarantees ohmic decay times of over the Hubble age, well above observational limits.

To explain the decay of the magnetic fields in millisecond pulsars additional mechanisms are required both for baryonic matter and quark matter. A not yet understood mechanism that buries the magnetic field seems the best candidate to describe the properties of compact stars in binary systems, as it allows the final field strength to depend on the duration of accretion.

In order to understand glitches, there has to be a mechanism able to store angular momentum that is quasi-periodically released to the surface. This is usually explained by the pinning of rotational vortices of the neutron superfluid to the crystal lattice of the surface. However, in quark matter this mechanism is not present. There might be a similar mechanism where angular momentum is stored and released between boundaries of one of the different states of colour superconductivity. However, the exact mechanism is not yet unclear.

Observational evidence that type II superconductivity does not occur in at least some of the compact stars is quite strong. The observed precession discussed in chapter 7 is incompatible with the presence of a type II superconductor. One solution is to assume a type I superconducting proton medium, a possibility that is still a matter of debate. The presence of quark matter also solves the precession problem, allowing for very long damping times on the precession.

Precession presents a very clear mechanism to study the interior of compact stars. In order to study the validity of the models, it is good to search for compact stars that show both glitches and free precession. The current models suggest that the combination is not possible, so if found this poses a severe problem for these models. Furthermore, if the number of events causing free precession is known, a statistical analysis of the phenomenon is possible. From this analysis the damping time can be obtained. The damping time gives direct insight in the structure of the matter in the interior of a compact star.

APPENDIX A

Derivation of the TOV equation

To derive the TOV-equation, it is useful to first consider the most general metric for a static, spherically symmetric system. This metric is

$$ds^{2} = -e^{2\alpha}dt^{2} + e^{2\beta}dr^{2} + r^{2}d\Omega^{2}$$
(A.1)

with α, β functions of r and the angular part given by

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{A.2}$$

Solutions for this metric in vacuum are the Schwarzschild solutions

$$ds^{2} = -c^{2} \left(1 - \frac{2GM}{c^{2}r} \right) dt^{2} + \left(1 - \frac{2GM}{c^{2}r} \right)^{-1} dr^{2} + r^{2} d\Omega^{2}$$
(A.3)

For the metric inside a star a vacuum cannot be assumed. In this case a solution to this metric from the Einstein equation with a nonzero energy momentum tensor has to be found

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{A.4}$$

with the Einstein tensor $G_{\mu\nu}$ defined as

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu}$$
(A.5)

were $R_{\mu\nu}$ is the Ricci tensor, contracted from the Riemann Tensor

$$R\mu\nu = R^{\lambda}_{\ \mu\lambda\nu} \tag{A.6}$$

The Riemann tensor is

$$R^{\lambda}_{\ \mu\lambda\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$
(A.7)

From the metric (equation A.1) the Christoffel symbols can be calculated

$$\Gamma_{tr}^{t} = \partial_{r}\alpha \qquad \Gamma_{tt}^{r} = e^{2(\alpha - \beta)}\partial_{r}\alpha \qquad \Gamma_{rr}^{r} = \partial_{r}\beta$$

$$\Gamma_{r\theta}^{\theta} = \frac{1}{r} \qquad \Gamma_{\theta\theta}^{r} = -re^{-2\beta} \qquad \Gamma_{r\phi}^{\phi} = \frac{1}{r}$$

$$\Gamma_{\phi\phi}^{r} = -re^{-2\beta}\sin^{2}\theta \qquad \Gamma_{\phi\phi}^{\theta} = \sin\theta\cos\theta \qquad \Gamma_{\theta\phi}^{\phi} = \frac{\cos\theta}{\sin\theta}$$

All symbols not specified are zero or related by symmetry relations. Using the previous declarations, one can calculate the Einstein tensors

$$G_{tt} = \frac{1}{r^2} e^{2(\alpha - \beta)} \left(2r\partial_r \beta - 1 + e^{2\beta} \right)$$

$$G_{rr} = \frac{1}{r^2} \left(2r\partial_r \alpha - 1 + e^{2\beta} \right)$$

$$G_{\theta\theta} = r^2 e^{2\beta} \left(\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta \frac{1}{r} (\partial_r \alpha - \partial_r \beta) \right)$$

$$G_{\phi\phi} = \sin^2 \theta G_{\theta\theta}$$
(A.8)

We assume that the star is made of a perfect fluid. A perfect fluid is a fluid with an isotropic pressure P and a rest-frame energy density ϵ . The energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\epsilon(r) + P(r))U_{\mu}U_{\nu} + P(r)g_{\mu\nu}$$
(A.9)

with the four-velocity U_{α} defined as $dx_{\alpha}/d\tau$, with τ the proper time. Searching for static solutions it is possible to choose the four-velocity in the timelike direction and normalise to $U^{\alpha}U_{\alpha} = -1$. Using this, the four-velocity becomes

$$U_{\mu} = (e^{\alpha}, 0, 0, 0) \tag{A.10}$$

The energy momentum tensor is then entirely diagonalised

$$T_{\mu\nu} = \mathbb{I} \cdot (e^{2\alpha} \epsilon, e^{2\beta} P, r^2 P, r^2 \sin^2(\theta) P)$$
(A.11)

Due to the diagonal form the Einstein equation splits into three equations

$$\frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\beta - 1 + e^{2\beta}\right) = 8\pi G\epsilon \tag{A.12}$$

$$\frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\alpha - 1 + e^{2\beta}\right) = 8\pi GP \tag{A.13}$$

$$e^{-2\beta} \left(\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta \frac{1}{r} (\partial_r \alpha - \partial_r \beta) \right) = 8\pi GP$$
(A.14)

Equation A.12 only depends on ϵ and β . It is therefore helpful to define a new function m(r) that depends only on r and β

$$m(r) = \frac{1}{2G}(r - re^{2\beta})$$
(A.15)

so that we can rewrite the metric (equation A.1) to

$$ds^{2} - e^{2\alpha(r)}dt^{2} + \left[1 - \frac{2Gm(r)}{r}\right]^{-1}dr^{2} + r^{2}d\Omega^{2}$$
(A.16)

and equation A.12 to

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \epsilon \tag{A.17}$$

This is the equation of mass conservation. Integrating this yields the total gravitational mass up to a radius r

$$m(r) = \int_0^r \epsilon r' r'^2 dr'$$
(A.18)

It must be, since at r = R, the outer radius of a star, the TOV metric should match the Schwarzschild metric. Note that the volume element of the integral is not the proper spatial volume element of curved space with a spatial metric γ_{ij}

$$\sqrt{\gamma}d^3x = e^\beta r^2 \sin\theta dr d\theta d\phi$$

Using this volume element we can calculate the total particle energy

$$E_{\text{particle}}(r) = 4\pi \int_0^r \frac{\epsilon(r')r'^2}{(1 - 2Gm(r')/r)^{1/2}} dr'$$
(A.19)

The difference $E_{\text{binding}} = E_{\text{particle}} - m(r)$ is the binding energy of the star.

Rewriting equation A.13 in terms of m(r) one obtains

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = \frac{Gm(r) + 4\pi Gr^3 P}{r(r - 2Gm(r))} \tag{A.20}$$

In order to solve the final equation obtained from the Einstein equation (equation A.14), it is convenient to use energy-momentum conservation: $\nabla_{\mu}T^{\mu\nu} = 0$. The nontrivial component is $\nu = r$, all in all the terms are:

$$\nabla_0 T^{0\nu} = \partial_r \alpha e^{2\beta} (P + \epsilon)$$

$$\nabla_1 T^{1\nu} = \partial_r (\beta e^{2\beta}) P + \partial_r \beta e^{2\beta} P$$

which combines to

$$(\epsilon + P)\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{\mathrm{d}P}{\mathrm{d}r} \tag{A.21}$$

Using this equation to eliminate α in equation A.20 yields the TOV equation

$$\frac{dp}{dr} = -\frac{(\epsilon + P)(Gm(r) + 4\pi Gr^3 P)}{r(r - 2Gm(r))}$$
(A.22)

APPENDIX B

Notation

Ζ	Partition function
N	Particle number
Ω	Thermodynamic potential
<i>x</i> , <i>t</i>	place and time coordinates
r	radial coordinate
M, M_{\odot}	Mass and solar mass
Т	Temperature
μ	Chemical potential
Р	Pressure
ϵ	Energy density
ho	Mass density for non-relativistic media
Ε	Energy
ϕ	Boson field, angle or phase
ψ	Fermion field
$\tau = it$	Wick rotated time
α	Unspecified coupling constant
α_s	QCD coupling constant
g	Gluon charge
S	The action
L, \mathcal{L}	Lagrangian, Lagrangian density
H, \mathcal{H}	Hamiltonian, Hamiltonian density

Table B.1: List of symbols used in this thesis

In this thesis I will use natural units, so $c = k = \hbar = 1$. The gravitational constant G will not be put equal to 1. When units are present, I will use SI units.

For vectors and sums, Greek indices generally indicate a vector in 4 dimensional spacetime and run from 0 to 4. Latin are mostly used for all other run over the three spatial dimensions or will run over the degrees of freedom of a system. An exception to this rule is when labelling quantum numbers of a colour conducting state. In this case *a*, *b* are the spin labels, *i*, *j* the flavour labels and α , β the colour labels. When indices occur twice in an equation, summation over these indices is implied.

When a calligraphic notation is used, the density of a quantity is meant. So L is a Lagrangian

Appendix B: Notation

while \mathcal{L} is a Lagrangian density. An exception is the energy density, for which I use the ϵ symbol throughout this thesis.

Furthermore, some astronomic notation will be used. A symbol with a $_{\odot}$ means the solar value of that symbol. For example, L_{\odot} is the solar luminosity. In a similar fashion, $_*$ means the value applicable to a specific star.

When not specified, logarithms are with base e, so $\log \equiv \ln$ in this thesis. I will only use \log .

Path integrals are indicated with a script \mathcal{D} . For example, the calculation of the expectation value of operator F in path integral notation is to be understood as

$$\langle F \rangle = \frac{\int \mathcal{D}\phi F(\phi) e^{iS[\phi]}}{\int \mathcal{D}\phi e^{iS[\phi]}} = \frac{\prod_n \int \mathcal{D}\phi_n F(\phi_n) e^{iS[\phi_n]}}{\prod_n \int \mathcal{D}\phi_n e^{iS[\phi_n]}}$$
(B.1)

where $S[\phi]$ is the action and ϕ_n is a decomposition of ϕ onto a base of orthonormal functions.

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Samenvatting



Figuur B.1: De kracht tussen quarks gedraagt zich als een soort veer. hoe verder de quarks gescheiden worden, hoe groter de kracht tussen de quarks wordt. Als de kracht te groot wordt 'breekt' de veer en ontstaan uit het vacuim twee nieuwe quarks (afbeelding van Lippert, 2005)

Zware sterren, sterren meer dan acht keer zo zwaar als onze zon, exploderen aan het eind van hun leven in een supernova-explosie. De buitenlagen van de ster worden met grote snelheid de ruimte ingeblazen terwijl de kern juist ineen stort tot een compacte ster. Deze compacte ster weegt meestal iets meer dan onze zon, maar is maar zo'n 20 kilometer groot. Dit betekent dat de materie in een compacte ster zeer dicht opeengepakt moet zijn. Om een idee te geven: als je één kubieke kilometer water tot de grootte van een suikerklontje perst, krijg je een vergelijkbare dichtheid.

Bij zulke hoge dichtheden gaat materie zich anders gedragen. Zo zijn atoomkernen niet meer stabiel, en kunnen de deeltjes in de atoomkernen (protonen en neutronen) zich vrij bewegen. Het zou ook kunnen dat de deeltjes waarvan protonen en neutronen gemaakt zijn (quarks) zich als vrije deeltjes gaan gedragen. Om dit te begrijpen moeten we kijken naar de eigenschappen van de kracht tussen de quarks, de sterke kracht. Quarks kunnen bij lage energieën niet vrij voorkomen. Lage energieën komen overeen met grote afstanden. De kracht tussen quarks werkt als een soort veer, en wordt sterk bij grote afstanden tussen de quarks. Dit is geïllustreerd in afbeelding B.1. Bij hoge energieën komen de 'veertjes' slap te staan en gaan de quarks zich gedragen als vrije deeltjes.

In zeer dichte materie moeten de deeltjes hele hoge energieën hebben. Dit komt omdat twee quarks niet dezelfde energie kunnen hebben vanwege het uitsluitingsprincipe. Als veel quarks erg dicht op elkaar geperst worden moet de gemiddelde quarkenergie dus toenemen. Voor zeer dichte materie kan de gemiddelde quark-energie zo hoog worden dat de binding tussen quarks zo zwak wordt dat ze niet langer opgesloten zitten in protonen en neutronen. In dat geval onstaan de net besproken quarksterren.

Voor mijn onderzoek heb ik gekeken naar de gevolgen van de aanwezigheid van quarkmaterie op waarneembare eigenschappen van deze compacte sterren. Als eerste heb ik gekeken naar het verband tussen massa en straal van deze sterren. Het blijkt dat dit verband erg anders is als de massastraalrelatie voor compacte sterren gemaakt van protonen en neutronen. Echter, voor massa's iets groter dan de massa van onze zon komen de diagrammen toch redelijk overeen. Dit zijn precies de

Samenvatting

massa's van waargenomen compacte sterren.

Een ander gevolg is de koppeling van de kern van de compacte ster aan het magneetveld. Het blijkt dat als de kern bestaat uit vrije quarks, er geen koppeling is tussen de kern en het magneetveld. Sinds kort zijn er waarnemingen die aangeven dat de rotatieas van sommige compacte sterren schommelt, dit heet precessie. We denken dat deze precessie alleen voor kan komen als er geen koppeling is tussen de kern en het magneetveld. Mijn conclusie is dat deze precessie een veelbelovende manier is om quarkmaterie op te sporen.

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