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SU(N) Yang-Mills Solutions with Constant Field Strength on T^4

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Abstract. We study for T^4 the class of solutions to the SU(N) Yang-Mills equations with constant field strength. The fluctuation spectrum is explicitly calculated in terms of generalized Riemann theta functions. We show that if these solutions are stable, they are necessarily (anti)-selfdual, in which case we verify the index theorem.

1. Introduction

Euclidean solutions to the classical equations of motion (instantons), play an important role in the nonperturbative analysis of gauge theories. For S^4 the most general solution is known now (ADHM construction [1]) using advanced mathematical results. It was hoped that they could be used to understand confinement, however Coleman's argument shows that instantons have no effect on the Wilson loop, which is used to measure the static quark-antiquark potential [3].

For the torus this argument is no longer valid, because twisted boundary conditions [2] force electric and magnetic flux through the box. Nevertheless one will encounter as always severe infrared problems. Physical quantities are expressed in terms of the running coupling constant, which for small box size L is proportional to $(-\ln L)^{-1/2}$. So if L increases, the running coupling constant increases and perturbation theory breaks down, not only in the perturbative but also in the instanton sectors. Including instantons however can give an earlier signal for the crossover. Moreover the analysis of Lüscher [4] shows that the energy of the ground state is independent of the electric flux $\mathbf{e}$ (the central sectors) to all orders in perturbation theory. Confinement would be signalled by an energy difference, between the ground state levels in each central sector, proportional to L. Since "twisted" instantons lift the degeneracy they might be crucial to detect this behaviour. This work is intended as a first step in that direction.

We will concentrate on gauge fields with constant curvature. Such soltuions were already considered some time ago by 't Hooft [5] for SU(N) on T^4 with twisted boundary conditions. Only if the sides of the box representing T^4 satisfy certain relations, these solutions are (anti)-selfdual. For small groups and arbitrary compact manifolds [including SU(2) (SO(3)) bundles over T^4] stable extrema of the action are (anti)-selfdual [6]. It is therefore no surprise that in SU(2) an explicit calculation shows that constant solutions which are stable are (anti)-selfdual (the reverse is obvious).

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For constant gauge fields on $\mathbb{R}^4$ this was already established by Leutwyler [7]. The boundary condition is given by requiring the action to differ by a finite amount from that of the constant background field. There is an infinite degeneracy in the fluctuation spectrum, which can be understood by the lack of control over the topology of the field configurations. On the hypercube (T^4) this problem is resolved, but complicated boundary conditions is the price we have to pay. For the boundary conditions that we choose (which are of the abelian type), the Riemann θ-functions provide a *natural* (from the mathematical point of view) solution to this problem. In the case of SU(2) each set of boundary conditions admitting constant solutions is gauge equivalent to an abelian set (Appendix B), but for SU(N), $N>2$, this is no longer true.

Constant field strength configurations also played an important role in the Copenhagen vacuum picture [8]. One starts from a constant chromomagnetic field in the z-direction. This is certainly not selfdual and thus unstable (unlike in the abelian case). Including the unstable mode yields an effective Higgs type potential. In analogy with the abelian Higgs model one minimizes the potential by assuming the formation of a lattice structure orthogonal to the z-direction, with size of the order of the inverse of the effective Higgs mass ($\simeq\sqrt{2g|B|}$, B the background chromomagnetic field strength). The reason for considering this type of vacuum is that, when including ("static") quantum fluctuations and minimizing with respect to the background magnetic field, the energy is smaller than that of the perturbative vacuum. Furthermore phenomenology indicates $\langle 0|B^2|0\rangle>0$, and the bag model gives similar results.

It has been shown by Ambjørn and Olesen [8] that their lattice structure is equivalent to imposing twisted boundary conditions on the unit cell in the x, y direction. In essence one considers gauge fields over the 2-torus T^2, the unstable mode is then given by the θ-function on T^2 (up to an exponential factor). It is therefore not really surprising that our analysis on T^4 is also based on θ-functions. Whereas the classical θ-function on the 2-torus (due to Jacobi) is well known, its generalization to a $2n$-torus (due to Riemann) is not.

This article will mainly concentrate on giving the necessary technical details.

As for the 2-dimensional case, there is a specific complex structure on T^4 in which the θ-functions are most easily expressed. This suggests looking for the most general (anti) selfdual solutions on T^4. It is amusing to note that for $\mathbb{C}P_1$ on T^2, the exact instanton solutions are expressed in terms of the Weierstrass σ-function [9a], which has a simple relation to the θ-function [9b]. Also the solution of Gürsey and Tze [10] is based on elliptic functions (the generalization of the Weierstrass ζ-function to quaternions). It seems to be a candidate for an instanton on T^4 with unit Chern number. However, one simply checks that the gauge invariant quantity $\mathrm{Tr}(F_{\mu\nu}^2)$ is not periodic and it can therefore not yield a solution on T^4. Explicitly their solution is of the form: (σ^a the Pauli matrices, η the 't Hooft η symbol [11]):

$$A_\mu=-\tfrac{1}{2}\sigma^a\bar{\eta}_{a\mu\nu}\partial^\nu \ln\varrho\,, \tag{1.1}$$

with

$$\varrho=\frac{1}{x^2}+\sum_{q\in L}\left\{\frac{1}{(x-q)^2}-\frac{1}{q^2}-\frac{2(x\cdot q)}{q^4}+\frac{x^2}{q^4}-\frac{4(x\cdot q)^2}{q^6}\right\}, \tag{1.2}$$

where L is the lattice (minus the origin) spanned by $q_\mu = \omega_{\mu\nu} n^\nu$, with $n \in \mathbb{Z}^4$ and $\det\omega \neq 0$. ϱ is pseudo periodic: $\varrho(x+q) = \varrho(x) + x \cdot a_q + b_q$, but

$$\mathrm{Tr}(F_{\mu\nu}^2) = -\Box\Box \ln\varrho, \tag{1.3}$$

with $\Box$ the four-dimensional Laplacian, is not periodic.

The organization of the rest of the paper is as follows. In Sect. 2 we will set up the notations and the fluctuation equations. We will use SU(N) gauge fields with twisted boundary conditions, which is equivalent to SO(N^2-1) fiber bundles over T^4 [12]. In Sect. 3 we will give the fluctuation spectrum. Self duality and stability of the solutions will be treated in Sect. 4, together with the index theorem. Section 5 gives a discussion of the results and an outlook for further research. Technicalities on the θ-functions are collected in an appendix.

2. The Solutions and Their Fluctuations

We will first review how to put gauge fields on the hypertorus. These gauge fields will be in the fundamental representation of SU(N) and are hermitian. The curvature or field strength is given by

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \tag{2.1}$$

The 4-torus will be labelled by a 4-dimensional hypercube $\{x \in \mathbb{R}^4 | 0 < x_\mu \leqq a_\mu\}$, with the standard metric of $\mathbb{R}^4$. It is not really essential that the hypercube is rectangular, but for physical applications [2] it is more convenient. From a mathematical point of view it is more appropriate to view T^4 as $\mathbb{R}^4$ modulo some 4-dimensional lattice L. So with

$$L = \{x \in \mathbb{R}^4 | x = n_\mu a^{(\mu)};\ n \in \mathbb{Z}^4\}, \tag{2.2}$$

we have $T^4 = \mathbb{R}^4/L$. Here $a^{(\mu)}$ is a vector in the μ-direction with length a_μ, they form the $\mathbb{Z}$-basis of the lattice L.

To define gauge fields on T^4, we take gauge potentials on $\mathbb{R}^4$ and demand that all gauge invariant quantities which can be formed out of them are periodic over L. This implies that the vector potential satisfies:

$$A_\lambda(x + a^{(\mu)}) = [\Omega_\mu(x)] A_\lambda(x) \equiv \Omega_\mu(x) A_\lambda(x) \Omega_\mu(x)^{-1} - i\Omega_\mu(x)\partial_\lambda \Omega_\mu^{-1}(x). \tag{2.3}$$

In the terminology of fiber bundles over the torus, $\Omega_\mu(x)$ is the elementary cocycle $E_{e(\mu)}(x)$ and in general we have

$$A_\lambda(x + n_\mu a^{(\mu)}) = E_n(x) A_\lambda(x), \tag{2.4}$$

where the cocycles have to satisfy the cocycle condition:

$$E_{n+m}(x) = E_n(x + m_\mu a^{(\mu)}) E_m(x). \tag{2.5}$$

This clearly implies $E_n(x+m)E_m(x) = E_m(x+n)E_n(x)$, and one easily shows that given the elementary coycles $E_{e(\mu)}$ this identity is satisfies if $E_n(x)$ is defined inductively by (2.5) and if:

$$E_{e(\mu)}(x + a^{(\nu)}) E_{e(\nu)}(x) = E_{e(\nu)}(x + a^{(\mu)}) E_{e(\mu)}(x). \tag{2.6}$$

Since $[Z\Omega_\mu]=[\Omega_\mu]$, for Z an element of the centre of SU(N), identity (2.6) implies:

$$\Omega_\mu(x+a^{(\nu)})\Omega_\nu(x)=Z_{\mu\nu}\Omega_\nu(x+a^{(\mu)})\Omega_\mu(x)\,. \tag{2.7}$$

$Z_{\mu\nu}$ is an element of the centre of SU(N) defining the twist of the bundle [2, 12]. It is a topological invariant, which is conveniently labelled by the twist tensor $n_{\mu\nu}$:

$$Z_{\mu\nu}=\exp(-2\pi i n_{\mu\nu}/N)\,, \tag{2.8}$$

n is an antisymmetric 4×4 matrix with entries in $\mathbb{Z}$ (mod N). Together with the Pontryagin index P_1, this specifies the topology of the fiber bundle:

$$P_1=-2NC_2\,,\qquad C_2=\frac{-1}{16\pi^2}\int_{T_4}\mathrm{Tr}(G_{\mu\nu}\tilde{G}_{\mu\nu})d_4x=\nu+\frac{N-1}{N}\mathrm{Pf}(n)\,, \tag{2.9}$$

where $\tilde{G}$ is the dual of G: $\tilde{G}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}G_{\alpha\beta}$. Pf($n$) is the Pfaffian of the twist tensor:

$$\mathrm{Pf}(n)=\tfrac{1}{4}n_{\mu\nu}\tilde{n}_{\mu\nu}=\tfrac{1}{8}n_{\mu\nu}n_{\alpha\beta}\varepsilon_{\mu\nu\alpha\beta}\,, \tag{2.10}$$

and ν is an integer, the "instanton number," see [12] for details. P_1 is always an even integer; C_2 is also integer if there is no twist ($Z_{\mu\nu}=1$ or equivalently $n_{\mu\nu}=0 \bmod N$).

We will restrict ourselves here to pure abelian boundary conditions; to be specific [12]:

$$\Omega_\mu(x)=\exp\left(-\frac{\pi i}{N}\sum_\nu\frac{n_{\mu\nu}x_\nu}{a_\nu}T\right), \tag{2.11}$$

with T the generator in SU(N) which "contains" the centre of SU(N):

$$T=\mathrm{diag}(1,\ldots,1,1-N)\,. \tag{2.12}$$

For these configurations $\nu=0$, or $C_2=\dfrac{N-1}{N}\mathrm{Pf}(n)$. An obvious solution to the Yang-Mills equations of motion:

$$D_\mu G^0_{\mu\nu}=\partial_\mu G^0_{\mu\nu}+i[A^0_\mu,G^0_{\mu\nu}]=0\,, \tag{2.13}$$

satisfying the boundary conditions (2.3), (2.11) is:

$$A^0_\mu=-\frac{\pi}{N}\sum_\nu\frac{n_{\mu\nu}x_\nu}{a_\mu a_\nu}T\,,\qquad G^0_{\mu\nu}=\frac{2\pi}{N}\frac{n_{\mu\nu}}{a_\mu a_\nu}T. \tag{2.14}$$

One of the reasons which enables us to explicitly compute the fluctuations around these solutions, is that they are sections of certain well-defined U(1) line bundles over T^4.

For this purpose we expand A_μ as follows:

$$A_\mu=A^0_\mu+\sum_{a=1}^{(N-1)^2}b^a_\mu T_a+\sum_{a=1}^{N-1}\sqrt{2}\,\mathrm{Re}(c^a_\mu\Sigma_a)\,, \tag{2.15}$$

where b is real, c is complex, T_a for $a=1$ up to $(N-1)^2-1$ are the generators of SU($N-1$) embedded in SU(N) in the upper left corner, $T_{(N-1)^2}=(2N(N-1))^{-1/2}T$ and $(\Sigma_a)_{kl}=\delta_{ka}\delta_{lN}$. The generators T_a supplemented with

$T_{(N-1)^2+2a+1}=\frac{1}{2}(\Sigma_a+\Sigma_a^+)$ and $T_{(N-1)^2+2a}=\frac{i}{2}(\Sigma_a-\Sigma_a^+)$, form the algebra of SU(N) with normalization $\mathrm{Tr}(T_a T_b)=\frac{1}{2}\delta_{ab}$. Using the commutation relations:

$$[T,\Sigma_a]=N\Sigma_a\,,\qquad [T,\Sigma_a^+]=-N\Sigma_a^+\,,\qquad [T,T_a]=0\,, \tag{2.16}$$

one finds that the boundary conditions on A_μ are satisfied if and only if:

$$b_\mu^a(x+a^{(\lambda)})=b_\mu^a(x)\,,\qquad c_\mu^a(x+a^{(\lambda)})=\exp\left(-\pi i\sum_\nu\frac{n_{\lambda\nu}x_\nu}{a_\nu}\right)c_\mu^a(x)\,. \tag{2.17}$$

So obviously b is a section of a trivial U(1) line-bundle and c a section of a non-trivial U(1) bundle with its first Chern class in $1-1$ relation with n. Again we can define for this bundle a cocycle $e_n(x)$:

$$\begin{aligned} c(x+n_\mu a^{(\mu)})&=e_n(x)c(x)\,,\\ e_{e^{(\mu)}}(x)&=\exp\left(-\pi i\sum_\nu\frac{n_{\mu\nu}x_\nu}{a_\nu}\right),\\ e_{n+m}(x)&=e_n(x+m_\mu a^{(\mu)})e_m(x)\,. \end{aligned} \tag{2.18}$$

Explicitly we have:

$$\begin{aligned} e_k(x)&=\exp\left(-\pi i\sum_{\mu,\nu}\frac{k_\mu n_{\mu\nu}x_\nu}{a_\nu}\right)\alpha(k)\,,\\ \alpha(k)&=\exp\left(\pi i\sum_{\mu<\nu}k_\mu n_{\mu\nu}k_\nu\right). \end{aligned} \tag{2.19}$$

We are now in a position to formulate the fluctuation equation, by expanding the action around the solution (2.14):

$$\begin{aligned} S=\tfrac{1}{2}\int d_4x\,\mathrm{Tr}(G_{\mu\nu}^2)=\tfrac{1}{2}\int d_4x\{&\mathrm{Tr}(G_{\mu\nu}^{0\,2})\\ &-\mathrm{Tr}(\delta A_\mu D_\nu^2\delta A_\mu-\delta A_\mu D_\mu D_\nu\delta A_\nu+2i\delta A_\mu[G_{\mu\nu}^0,\delta A_\nu])\\ &+2i\,\mathrm{Tr}(\delta A_\mu D_\nu[\delta A_\mu,\delta A_\nu])+\tfrac{1}{2}\mathrm{Tr}((i[\delta A_\mu,\delta A_\nu])^2)\}\,, \end{aligned} \tag{2.20}$$

where $A_\mu=A_\mu^0+\delta A_\mu$ and $D_\mu=\partial_\mu+i[A_\mu^0,\cdot]$. Furthermore we introduce as usual background gauge fixing, $D_\mu\delta A_\mu=0$, so that in general the action including the Faddeev-Popov fields is given by [11]:

$$S=\int d_4x\{\tfrac{1}{2}\mathrm{Tr}(G_{\mu\nu}^0)^2+\tfrac{1}{2}\mathrm{Tr}(\delta A_\mu M_A^{\mu\nu}\delta A_\nu)+\mathrm{Tr}(\psi^+M_{gh}\psi)+\mathcal{O}(\delta A^3)\}\,, \tag{2.21}$$

with

$$M_A^{\mu\nu}=-\delta_{\mu\nu}D_\lambda^2-2i[G_{\mu\nu}^0,\cdot]\,,\qquad M_{gh}=-D_\lambda^2\,. \tag{2.22}$$

If we substitute for δA_μ the expression in (2.15) and expand the ghost fields according to:

$$\psi=\sum_{a=1}^{(N-1)^2}\psi^a T_a+\sum_{a=1}^{N-1}\sqrt{2}(\phi^a\Sigma_a+\{\phi^{a+N-1}\Sigma_a\}^\dagger) \tag{2.23}$$

with the appropriate boundary conditions:

$$\psi^a(x+n_\mu a^{(\mu)})=\psi^a(x)\,, \qquad \phi^a(x+n_\mu a^{(\mu)})=e_n(x)\phi^a(x)\,, \tag{2.24}$$

then the action takes the form:

$$\begin{aligned} S=S_0+\int d_4x\{&\tfrac{1}{2}b^a_\mu(M_0\delta^{\mu\nu}\delta_{ab})b^b_\nu+c^{a*}_\mu(M_n\delta^{\mu\nu}-4\pi iF^{\mu\nu})\delta_{ab}c^b_\nu+\psi^{a*}(M_0\delta_{ab})\psi^b\\ &+\phi^{a*}(M_n\delta_{ab})\phi^b+\mathcal{O}((\delta A_\mu)^3)\}\,, \end{aligned} \tag{2.25}$$

with

$$\begin{aligned} S_0&=\tfrac{1}{2}\int d_4x\,\mathrm{Tr}(G^{0\,2}_{\mu\nu})=\frac{2\pi^2(N-1)}{N}F_{\mu\nu}{}^2V,\\ F_{\mu\nu}&=\frac{n_{\mu\nu}}{a_\mu a_\nu}, \qquad V=\prod_{\mu=1}^{4}a_\mu\,,\\ M_0&=\left(\frac{1}{i}\partial_\mu\right)^2,\\ M_n&=\left(\frac{1}{i}\partial_\mu-\pi F_{\mu\nu}x_\nu\right)^2. \end{aligned} \tag{2.26}$$

Before we will construct the spectrum of these operators, let us briefly discuss the generalization for an arbitrary gauge group H. For convenience we will restrict ourselves to a simple and simply connected group [like SU(N)]. Generalization to semi-simple, multiple connected groups and details can be found with the help of Goddard et al. [13] and Humphreys [14] (see also [20]). One can always choose the following basis of the Lie-algebra of H:

$$T_1,\ldots,T_r,E_{\pm\alpha}{}^{(1)},\ldots,E_{\pm\alpha}{}^{(s)}\,, \tag{2.27}$$

where r is the rank of H (for SU(N), $r=N-1$, $s=\frac{1}{2}N(N-1)$, $E_{-\alpha}=E^\dagger_\alpha$, $T^\dagger=T$) and

$$[T_i,E_\alpha]=\alpha_iE_\alpha\,, \qquad [T_i,T_j]=0\,. \tag{2.28}$$

The $\alpha^{(i)}$ are called the roots and they span an r-dimensional space, whose metric is given by the Killing form $\kappa(X,Y)=\mathrm{Tr}_{ad}(XY)$ restricted to the maximal torus (or Cartan subalgebra, spanned by $T_1,\ldots,T_r$)

One can choose normalizations such that this metric is δ_{ij}:

$$\mathrm{Tr}_{ad}(T_iT_j)=2N\,\mathrm{Tr}(T_iT_j)=\delta_{ij}\,, \qquad \sum_\alpha \alpha_i\alpha_j=\delta_{ij}\,. \tag{2.29}$$

The abelian boundary conditions are now

$$\Omega_\mu(x)=\exp\left(-\pi i\sum_\nu\frac{n^k_{\mu\nu}x_\nu}{a_\nu}T_k\right), \tag{2.30}$$

and the cocycle conditions imply

$$\exp(2\pi in^k_{\mu\nu}T_k)\in Z(H)\,. \tag{2.31}$$

Using (2.28) one easily finds that this is true if and only if:

$$\exp(2\pi i n^k_{\mu\nu}\alpha_k)=1 \quad \text{or} \quad n_{\mu\nu}\cdot\alpha\in\mathbb{Z} \tag{2.32}$$

for all roots α. So for each μ, ν $\frac{n_{\mu\nu}}{2N}$ is a weight for the "dual" of H [13], which is nothing but the Dirac quantization condition [Eq. (2.31) is the condition for the "single valuedness"]. The solutions of the Yang-Mills equations of motion are:

$$A^0_\mu=-\sum_\nu \frac{n^k_{\mu\nu}x_\nu}{a_\mu a_\nu}T_k\,, \qquad G^0_{\mu\nu}=2\frac{n^k_{\mu\nu}}{a_\mu a_\nu}T_k\,. \tag{2.33}$$

Expanding around these solutions to implement the boundary conditions on the fluctuations we find:

$$A_\mu=A^0_\mu+b^k_\mu T_k+\frac{1}{\sqrt{2}}c^\alpha_\mu E_\alpha\,, \qquad b(x+a^{(\mu)})=b(x)\,,$$

$$c^\alpha_\mu(x+a^{(\lambda)})=\exp\left(-\pi i\sum_\nu\frac{\alpha\cdot n\lambda_\nu x_\nu}{a_\nu}\right)c^\alpha_\mu(x)\,. \tag{2.34}$$

And (2.32) again implies that we have sections of line bundles with a first Chern-index $\alpha\cdot n_{\mu\nu}$. Furthermore we have

$$P_1=2\sum_k \mathrm{Pf}(n^k_{\mu\nu})=2\sum_\alpha \mathrm{Pf}(\alpha\cdot n_{\mu\nu})\,, \tag{2.35}$$

which is again twice an integer.

3. The Fluctuation Spectrum

We will introduce suitable complex coordinates, in which the fluctuation equations obtain an especially simple form. This is also suggested by Leutwyler's [7] analysis, but the amazing thing is that the boundary conditions are compatible with this choice and the whole analysis becomes canonical if one realizes that F [see (2.26)] introduces a positive definite hermitian form H. To be specific we first introduce coordinates:

$$\hat{x}=Sx\,, \qquad S\in\mathcal{O}(4)\,, \tag{3.1}$$

which brings F in the standard form:

$$SFS^t=\hat{F}=\begin{pmatrix} \emptyset & \begin{matrix} f_1 & 0\\ 0 & f_2\end{matrix} \\ \begin{matrix} -f_1 & 0\\ 0 & -f_2\end{matrix} & \emptyset \end{pmatrix}, \qquad f_1\geqq f_2\geqq 0\,. \tag{3.2}$$

The complex coordinates will be chosen according to

$$z(x)=(z_1,z_2)=\frac{1}{\sqrt{2}}(\hat{x}_1-i\hat{x}_3,\hat{x}_2-i\hat{x}_4)\,, \tag{3.3}$$

and the positive definite hermitian form will be given by

$$H(z,w)=2(z_1f_1\bar{w}_1+z_2f_2\bar{w}_2)=w^\dagger hz\,, \tag{3.4}$$

with $h=2\,\mathrm{diag}(f_1, f_2)$[1]. If we furthermore define:

$$(A_{z_1}, A_{z_2}) = \frac{1}{\sqrt{2}}(\hat{A}_1 + i\hat{A}_3, \hat{A}_2 + i\hat{A}_4), \tag{3.5}$$

$$(\partial_{z_1}, \partial_{z_2}) = \frac{1}{\sqrt{2}}(\partial_{\hat{x}_1} + i\partial_{\hat{x}_3}, \partial_{\hat{x}_2} + i\partial_{\hat{x}_4}), \tag{3.6}$$

$$a=(a_1, a_2); \qquad a_k = \frac{1}{i}\frac{\partial}{\partial \bar{z}_k} - \frac{i\pi}{2} h_{kl} z_l, \tag{3.7}$$

we find [see (2.26)]

$$M_n = \{a, a^\dagger\}, \tag{3.8}$$

$$c_\mu^{a*}(M_n\delta_{\mu\nu} - 4\pi i F_{\mu\nu})c_\nu^a = (c_{z_k}^a)^* (M_n\delta_{kl} - 2\pi h_{kl})c_{z_l}^a + (c_{\bar{z}_k}^a)^* (M_n\delta_{kl} + 2\pi h_{kl})c_{\bar{z}_l}^a . \tag{3.9}$$

Clearly the operators a_k and $a_k^\dagger$ are annihilation and creation operators, since:

$$[a_k, a_l^\dagger] = \pi h_{kl} = 2\pi f_k \delta_{kl} . \tag{3.10}$$

Therefore the spectrum of M_n is given by[2]

$$\lambda_m = 2\pi \sum_{k=1}^{2} (2m_k + 1) f_k , \qquad \chi_m = \frac{(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2}}{\sqrt{m_1! m_2!}} \chi_0 , \tag{3.11}$$

where χ_0 is the ground state, uniquely determined by

$$a\chi_0 = \frac{1}{i}\left(\frac{\partial}{\partial \bar{z}_k} + \frac{\pi}{2} h_{kl} z_l\right)\chi_0 = 0 , \tag{3.12}$$

and the boundary conditions. Explicitly

$$\chi_0 = e^{-\frac{\pi}{2} H(z,z)} f(z) , \tag{3.13}$$

where $f(z)$ is any holomorphic function, such that χ_0 satisfies the boundary conditions.

Let us define unit vectors $e_\mu^{(z_k)}$ and $e_\mu^{(\bar{z}_k)}$, with the properties $e^{(\bar{z}_k)} = (e^{(z_k)})^*$ and $e_{z_l}^{(z_k)} = \delta_{kl}$, $e_{\bar{z}_l}^{(z_k)} = 0$. Explicitly with respect to the $\hat{x}$ basis in (3.1) we have

$$\hat{e}^{(z_1)} = \frac{1}{\sqrt{2}}(1, 0, -i, 0) = [\hat{e}^{(\bar{z}_1)}]^* , \qquad \hat{e}^{(z_2)} = \frac{1}{\sqrt{2}}(0, 1, 0, -i) = [\hat{e}^{(\bar{z}_2)}]^* . \tag{3.14}$$

Then the spectrum of $M_n\delta_{\mu\nu} - 4\pi i F_{\mu\nu}$ is also easily determined:

$$\begin{aligned} \lambda_{m,k}^- &= \lambda_m - 4\pi f_k , \qquad \chi_{m,k}^- = \chi_m e^{(z_k)} \\ \lambda_{m,k}^+ &= \lambda_m + 4\pi f_k , \qquad \chi_{m,k}^+ = \chi_m e^{(\bar{z}_k)} . \end{aligned} \tag{3.15}$$

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1 If one prefers to work with $H(z,w) = z^\dagger h w$ one should replace (3.3) by $z = (\hat{x}_1 + i\hat{x}_3, \hat{x}_2 + i\hat{x}_4)/\sqrt{2}$

2 For the time being we assume $f_k \neq 0$, so H non-degenerate

For Leutwyler's analysis [7] the boundary conditions are that χ is square integrabel, but this means that $f(z)$ in (3.13) can be any holomorphic function, whence the infinite degeneracy. In our case f has to satisfy the boundary conditions $f(z+q)=u_q(z)f(z)$, with $u_q(z)$ the appropriate cocycle. In order for (3.13) to admit a non-trivial solution the coycle u_q must necessarily be a holomorphic function. That this turns out to be the case is not really a surprise since the spectrum of a bounded hermitian operator can never be empty. Thus we look for holomorphic sections on a complex torus: $T^4=\mathbb{C}^2/L$. The canonical objects for these are the (Riemann) θ-functions. For the theory of θ-functions we refer to Igusa [15], whose notations we will roughly follow.

The crucial thing for the complex structure on $\mathbb{C}^2/L$ with hermitian form H to admit θ-functions as holomorphic sections is that

$$E(z,w)=\operatorname{Im}H(z,w) \tag{3.16}$$

restricted to the lattice $L=\{k_\mu\zeta^{(\mu)}|k\in Z^4,\ \zeta^{(\mu)}=z(a^{(\mu)})\}$ takes values in the integers. With respect to the real basis one can simply express the hermitian form explicitly:

$$H(z(x),z(y))=x_\mu|F|_{\mu\nu}y_\nu+ix_\mu F_{\mu\nu}y_\nu\,, \tag{3.17}$$

where $|F|=(-F^2)^{1/2}$. To prove this, first express things in the $\hat{x}$ basis of Eq. (3.1), where $(-\hat{F}^2)^{1/2}=\operatorname{diag}(f_1,f_2,f_1,f_2)$. So E is nothing but F and $E/L=n$:

$$E(\zeta^{(\mu)},\zeta^{(\nu)})=n_{\mu\nu}\,,\qquad \zeta^{(\mu)}\equiv z(a^{(\mu)})\,. \tag{3.18}$$

The elementary cocycle of (2.18) can also be expressed in terms of E and z:

$$e_{e^{(\mu)}}(x)=\exp(-\pi iE(\zeta^{(\mu)},z(x)))\,. \tag{3.19}$$

The boundary condition for χ_0 is therefore

$$\chi_0(z+n_\mu\zeta^{(\mu)})=\exp(-\pi iE(n_\mu\zeta^{(\mu)},z))\alpha(n)\chi_0(z)\,. \tag{3.20}$$

Note that, as it should be, χ_m as defined in (3.11) satisfies the same boundary conditions. For this one uses the property

$$a^\dagger(z+q)e_q(z)=e_q(z)a^\dagger(z),\ q\in L\,. \tag{3.21}$$

To make the holomorphic structure visible we finally write down the boundary conditions on $f(z)$:

$$\begin{aligned} f(z+q)&=\alpha(q)\exp\left(\frac{\pi}{2}H(q,q)+\pi H(z,q)\right)f(z)\\ &\equiv u_q(z)\,f(z)\,,\qquad q\in L\,, \end{aligned} \tag{3.22}$$

where $\alpha(q)$ is defined in (2.19). It satisfies:

$$\alpha(q+r)=\alpha(q)\alpha(r)\exp(\pi iE(q,r))\,. \tag{3.23}$$

The holomorphic cocycle u_q is also called an automorphy factor. Any α defined on L satisfying (3.23) and having values in U(1) is called a second degree character strongly associated with E. Then all functions defined by (3.22) are called θ-functions of type (H,α); they form a complex linear space $L(H,\alpha)$ of dimension $|\mathrm{Pf}(n)|$, see [15, Chap. II] for details, some of which are collected in the appendix.

We assumed that H is non-degenerate ($f_1, f_2 \neq 0$); this is not really necessary. If $f_2=0, f_1 \neq 0$, (3.11) still generates part of the spectrum ($m_2 \equiv 0$), especially the $m=0$ modes are still given by θ-functions. The dimension of $L(H,\alpha)$ is in this case the greatest common divisor of the entries of n [g.c.d. $(n_{\mu\nu})$], see the appendix for details. There is one θ-function in $L(H,\alpha)$ which has an especially simple form (for H non-degenerate):

$$f_0(z) = \sum_{q \in L} \alpha(q) \exp\left(H(z,q) - \frac{\pi}{2} H(q,q) \right). \tag{3.24}$$

We will call it the intrinsic θ-function [16]. That the analysis of the spectrum of M_n is indeed canonical can be found in [17] and references therein, where the spectrum of the Laplace-Beltrami operator (M_n up to a constant) is constructed.

Let us finally collect our results to give explicitly the eigenfunctions and eigenvalues for the operators in (2.22):

	$M_A \delta A = \lambda \delta A;\ \delta A = \sqrt{2}\,\mathrm{Re}\,\phi$		$M_{gh}\psi = \lambda\psi$	
	ϕ	λ	ψ	λ
$f_i \neq 0$	$e^{(\mu)}\phi^{(p,a)}$	$\sum_\mu \left(\frac{2\pi p_\mu}{a_\mu}\right)^2$	$\phi^{(p,a)}$	$\sum_\mu \left(\frac{2\pi p_\mu}{a_\mu}\right)^2$
	$e^{(z_k)}\phi^{(m,r,b)}$ $ie^{(z_k)}\phi^{(m,r,b)}$	$2\pi\left(\sum_{i=1}^{2}(2m_i+1)f_i - 2f_k\right)$	$\phi^{(m,r,b)}$	$2\pi \sum_{i=1}^{2}(2m_i+1)f_i$
	$e^{(\bar{z}_k)}\phi^{(m,r,b)}$ $ie^{(\bar{z}_k)}\phi^{(m,r,b)}$	$2\pi\left(\sum_{i=1}^{2}(2m_i+1)f_i + 2f_k\right)$	$\phi^{(m,r,b+N-1)}$	$2\pi \sum_{i=1}^{2}(2m_i+1)f_i$
$f_i = 0$	$e^{(\mu)}\phi^{(p,c)}$	$\sum_u \left(\frac{2\pi p_\mu}{a_\mu}\right)^2$	$\phi^{(p,c)}$	$\sum_\mu \left(\frac{2\pi p_\mu}{a_\mu}\right)^2$
$f_2 = 0, f_1 \neq 0$ $\lambda < 0$ *only*	$e^{(z_1)}\phi^{(0,r,a)}$ $ie^{(z_1)}\phi^{(0,r,a)}$	$\lambda = -2\pi f_1$		

(3.25)

where a, b, c, k, m, p, and r have the range:

$$a = 1, \ldots, (N-1)^2; \quad b = 1, \ldots, (N-1); \quad c = 1, \ldots, (N^2-1);$$
$$k = 1, 2; \quad m \in \mathbb{N}^2; \quad p \in \mathbb{Z}^4;$$
$$r_1 = 0 \ldots (e_1 - 1); \quad r_2 = 0 \ldots (e_2 - 1).$$

We used the following definitions for the normalized eigenfunctions ($\Sigma_a \equiv \Sigma_{a+1-N}$ for $a > N-1$)

$$\begin{aligned} \phi^{(p,a)} &= \frac{1}{\sqrt{V}} e^{2\pi i p_\mu x_\mu / a_\mu} \cdot T_a, \\ \phi^{(m,r,a)} &= \frac{(a_1^+)^{m_1}(a_2^+)^{m_2}}{\sqrt{m_1! m_2!}} \chi_r \Sigma_a, \\ e_\nu^{(\mu)} = \delta_{\mu\nu}; \quad e_\mu^{(z_1)} &= \frac{1}{\sqrt{2}}(S_{1\mu} - iS_{3\mu}), \\ e_\mu^{(z_2)} &= \frac{1}{\sqrt{2}}(S_{2\mu} - iS_{4\mu}). \end{aligned} \tag{3.26}$$

The χ_r form an orthogonal basis for $a\chi=0$ (3.12) and are defined in the appendix (A.23).

The generalization to the solutions in (2.33) is straightforward and will therefore not be worked out in detail.

4. Stability and Selfduality

Using the results of the previous section we find $2(N-1)|\mathrm{Pf}(n)|$ negative eigenvalues ($\lambda=2\pi(f_2-f_1)$) if and only if $f_1 \neq f_2$ (by construction we choose $f_1 \geqq f_2 \geqq 0$). The duality equation is invariant under $\mathcal{O}(4)$ transformations S. If $\det S=-1$ self and antiselfdual configurations are interchanged. Thus the constant field strength solution is (anti) selfdual iff $f_1=f_2$. (In the $\hat{x}$ coordinates the solution is always antiselfdual). We can conclude from this that the constant solutions are stable if and only if they are (anti)-selfdual. If $f_1 \neq f_2$, the number of negative (and zero) modes is consistent with the lower bound derived by Taubes, Theorem 3.8 [23]. Selfduality imposes for a given twist tensor $n_{\mu\nu}$ constraints on the sides a_μ of the euclidean box which parametrizes the torus (or equivalently the a_μ determine the scale of the coordinates). But there is also a constraint on $n_{\mu\nu}$ itself to admit an (anti)-selfdual solution. It is necessary and sufficient that $\mathrm{sign}(n_{\mu\nu})$ is self or anti-selfdual. If $n_{\mu\nu} \neq 0$ for all $\mu \neq \nu$, a_μ is fixed up to an overall scale (no summation):

$$a_\mu = \left|\frac{n_{\alpha\mu}n_{\mu\beta}}{n_{\alpha\nu}n_{\nu\beta}}\right|^{1/2} a_\nu\,, \tag{4.1}$$

where μ, ν, α, β are all different. In general the number of undetermined scales is one more than the number of zeros in (n_{14}, n_{24}, n_{34}). These are similar conditions as found by 't Hooft [5]. His boundary conditions are however more complicated for $N \neq 2$. But we are confident that a similar analysis can be performed. One needs a generalization of the θ-function, where the bicharacters $\alpha(k)$ take their values in SU(N).

We will now use the symmetries of the solution to describe the degeneracies. With (2.14), (3.2), and (3.5) the solution in complex coordinates is given by:

$$A^0_{z_i} = \frac{i\pi}{N} f_i \bar{z}_i T\,. \tag{4.2}$$

The connection 1-form $A=A_\mu dx_\mu$ is given by $A=A_{z_i}dz_i+A_{\bar{z}_i}d\bar{z}_i$. The solution is therefore obviously invariant under 1) the unitary coordinate transformation $z\to Uz$ which leaves h invariant, and 2) the global gauge transformations which commute with the holonomy group of (4.2)[3]. These gauge transformations form the group generated by T_a for $a=1...(N-1)^2$ with covering U($N-1$). The unitary transformations leaving h invariant form the group U(1) × U(1) for $f_1 \neq f_2$ and U(2) for $f_1=f_2$. In the real $\hat{x}$ coordinates the group U(2) is SO(4)∩SP(2). Explicitly it is generated by

$$\begin{aligned} \tau_0&=\sigma_2\otimes 1_2\,, & \tau_1&=\sigma_2\otimes\sigma_1\,,\\ \tau_2&=-1_2\otimes\sigma_2\,, & \tau_3&=-\sigma_2\otimes\sigma_3\,, \end{aligned} \tag{4.3}$$

3 The holonomy group of A_μ is given by $P\exp\left(i\int_c A_\mu dx_\mu\right)$, where c runs over all closed loops, with a fixed base point, see [18, 20]

where the tensor product $\otimes$ is such that $\hat{F}=-if\sigma_2\otimes\sigma_3$. τ_0 and τ_3 generate the subgroup $U(1)\times U(1)$. The fluctuations have to be representations of these groups, which explains most of the degeneracies in the eigenvalues.

Assuming from now on selfduality ($f_1=f_2\equiv f$) we find the following eigenvalues (λ) and degeneracies (μ):

	M_A		M_{gh}	
	λ	μ	λ	μ
$f\neq 0$	$4\pi f\lvert k\rvert$	$4(N-1)\lvert\hat{k}\cdot \mathrm{Pf}(n)\rvert$	$4\pi f\lvert k\rvert;\, k\neq 0$	$(N-1)\lvert k\cdot \mathrm{Pf}(n)\rvert$
	$\sum_\mu\left(\frac{2\pi p_\mu}{a_\mu}\right)^2$	$4(N-1)^2$	$\sum_\mu\left(\frac{2\pi p_\mu}{a_\mu}\right)^2$	$(N-1)^2$
$f=0$	$\sum_\mu\left(\frac{2\pi p_\mu}{a_\mu}\right)^2$	$4(N^2-1)$	$\sum_\mu\left(\frac{2\pi p_\mu}{a_\mu}\right)^2$	(N^2-1)

(4.4)

where $k\in\mathbb{Z}$ and $p\in\mathbb{Z}^4$ and $\hat{k}=k+\delta_{k,0}$.

From now on we will only concentrate on the zero-modes. Explicitly they are given by [see (3.25), (3.26)]

	M_A	M_{gh}
$f\neq 0$	$\sqrt{2}\,\mathrm{Re}(e^{(z_k)}\chi_r\Sigma_b)$ $\sqrt{2}\,\mathrm{Im}(e^{(z_k)}\chi_r\Sigma_b)$ $V^{-1/2}e^{(\mu)}T_a$	$V^{-1/2}T_a$
$f=0$	$V^{-1/2}e^{(\mu)}T_c$	$V^{-1/2}T_c$

(4.5)

where $k=1,2$; $a=1\ldots(N-1)^2$; $b=1\ldots(N-1)$; $c=1\ldots(N^2-1)$; $r_1=0\ldots(e_1-1)$; $r_2=0\ldots(e_2-1)$ and $\mu=1,2,3$ or 4.

Before we discuss which modes are physical we work out the index of the appropriate operator to check the index theorem, which has been calculated in the literature for arbitrary base manifold and gauge group [18, 19]. If T is the following operator (in the selfdual sector):

$$T:\delta A_\mu\rightarrow(-\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}D_{[\alpha}\delta A_{\beta]}+D_{[\mu}\delta A_{\nu]},D_\mu\delta A_\mu)\,, \tag{4.6}$$

and if $f_{\mu\nu}$ is an anti-selfdual 2-form with values in the Lie algebra and g also takes values in this algebra, we can define the inner products:

$$\begin{aligned}(\delta A,\delta A')&=\mathrm{Tr}\int_{T_4}(\delta A_\mu^\dagger\delta A'^\mu)\,,\\((f,g),(f'g'))&=\mathrm{Tr}\int_{T_4}(\tfrac{1}{4}f_{\mu\nu}^\dagger f'^{\mu\nu}+g^\dagger g')\,.\end{aligned}\tag{4.7}$$

With respect to these inner products we can find the adjoint of T:

$$T^\dagger:(f,g)\rightarrow-(D_\mu f_{\mu\nu}+D_\nu g)\,. \tag{4.8}$$

After some calculations one finds

$$T^{\dagger}T=M_A, \tag{4.9}$$

and

$$TT^{\dagger}(f,g)=(-D^2f,-D^2g)=M_{gh}(f,g). \tag{4.10}$$

On T^4, g and f satisfy the same boundary conditions as δA. So we find:

$$\begin{aligned}\operatorname{index} T &= \dim\ker T - \dim\ker T^{\dagger} = \dim\ker T^{\dagger}T - \dim\ker TT^{\dagger}\\ &= \dim\ker M_A - 4\dim\ker M_{gh},\end{aligned} \tag{4.11}$$

and with the results from Eq. (4.5) this implies:

$$\operatorname{index} T = 4(N-1)\,|\mathrm{Pf}(n)|, \tag{4.12}$$

which is exactly the number of *non-constant* zero modes. This corresponds with the general result for T^4 [18], which easily extends to the twisted case too:

$$\operatorname{index} T = 4N|C_2| = 2|P_1|. \tag{4.13}$$

To see which modes are physical we first consider the topologically trivial case with zero twist ($n_{\mu\nu}=0$) and periodic boundary conditions. There we can easily write down the most general solution [4]:

$$A_\mu = \frac{1}{a_\mu}\operatorname{diag}(\varphi_\mu^{(1)},\ldots,\varphi_\mu^{(N)}), \qquad \sum_{i=1}^{N}\varphi_\mu^{(i)}=0, \tag{4.14}$$

where two solutions A and A' of this form are gauge equivalent (allowing *periodic* gauge transformations only) if and only if for all μ and i

$$\varphi_\mu'^{(i)} = \varphi_\mu^{(\sigma(i))}(\mathrm{mod}\,2\pi), \tag{4.15}$$

with σ some N-permutation. All solutions are gauge equivalent to the above ones. Wilson loops which wind around the torus should be left invariant under gauge transformations; we therefore allow for periodic gauge transformations only[4]. For almost all solutions (4.14), i.e. $\varphi_\mu^{(i)} \neq \varphi_\mu^{(j)}(\mathrm{mod}\,2\pi)$ for all $i \neq j$, M_A has $4(N-1)$ and M_{gh} has $(N-1)$ zero modes. Those of M_A are to be treated as collective coordinates. Those of M_{gh} are a consequence of the constant gauge transformations which leave A invariant; they are deleted from the spectrum of M_{gh} (see [18] for details, a factor (volume $H_A)^{-1}$ enters the path integral, where H_A is the group which leaves A invariant).

In the topologically nontrivial case with non-zero twist we do not know the most general solution, but we can at least exhibit the most general *abelian* solution:

$$A_\mu = -\frac{\pi}{N}F_{\mu\nu}x_\nu T + \frac{1}{a_\mu}\operatorname{diag}(\varphi_\mu^{(1)},\ldots,\varphi_\mu^{(N)}), \qquad \sum_{i=1}^{N}\varphi_\mu^{(i)}=0, \tag{4.16}$$

where again two solutions are gauge equivalent if and only if for all μ and i:

$$\varphi_\mu'^{(i)} = \varphi_\mu^{(\sigma(i))}(\mathrm{mod}\,2\pi). \tag{4.17}$$

---

4 Equivalently, allowed gauge transformations (Ω) have to satisfy $\Omega_\mu(x) = \Omega(x+a^{(\mu)})\Omega_\mu(x)\Omega(x)^{-1}\,\mathrm{mod}\,Z_N$, therefore leaving the boundary conditions invariant

However in this case the permutation σ should leave N fixed[5]. This constraint on σ comes from the fact that the allowed gauge transformations have to be periodic *and* leave T fixed. We leave it to the reader to show that also for the solutions in Eq. (4.16) the fluctuation spectrum can be found explicitly. In particular for almost all solutions (4.16), [i.e. $\varphi_\mu^{(i)} \neq \varphi_\mu^{(j)} (\mathrm{mod}\, 2\pi)$ for all $i \neq j$], M_A has $4(N-1)\,(|\mathrm{Pf}(n)|+1)$ and M_{gh} has $(N-1)$ zero-modes. The non-constant zero-modes of M_A are again given by θ-functions, this time with shifted arguments and modified bicharacters α.

Let us digress to the case of SU(2). We want to determine the number of parameters for an (anti)-selfdual solution if our abelian solution is not (anti)-selfdual. The number of zero-modes in M_A cannot be smaller than the index of T. If it is larger, $\dim\ker M_{gh}$ is necessarily non-zero. So let us investigate solutions ψ [with values in the algebra of SU(2)] of $D_\mu \psi = 0$ (if there are nontrivial solutions the kernel of M_{gh} is not zero). This obviously implies $[D_\mu, D_\nu]\psi = 0$ or equivalently:

$$[G_{\mu\nu}, \psi] = 0\,. \tag{4.18}$$

We find therefore in SU(2),

$$G_{\mu\nu}(x) = f_{\mu\nu}(x)\psi(x)\,, \tag{4.19}$$

where $f_{\mu\nu}(x)$ is an (anti)-selfdual 2-form with real values. The condition that $G_{\mu\nu}$ satisfies the Bianchi identity and the Yang-Mills equations of motion imply

$$\partial_\mu f_{\mu\nu} = 0\,, \qquad \partial_{[\mu} f_{\mu\nu]} = 0\,. \tag{4.20}$$

So $f_{\mu\nu}$ is the field strength of an abelian gauge field b_μ,

$$f_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu\,. \tag{4.21}$$

Obviously the nonabelian gauge potential

$$A_\mu = b_\mu \psi \tag{4.22}$$

gives rise to (4.19). The general form of A_μ is therefore a gauge transformation Ω of (4.22) which leaves (4.19) fixed, so $\Omega\psi\Omega^{-1} = \psi$. This implies that $\Omega\partial_\mu\Omega^{-1}$ is proportional to ψ and thus it only changes b_μ by an abelian gauge transformation and A_μ is still of the form (4.22). Finally we have to implement the boundary conditions with [see (2.11)]

$$\Omega_\mu = \exp\left(\frac{-\pi i}{2} \sum_\nu \frac{n_{\mu\nu} x_\nu}{a_\nu} \sigma_3\right). \tag{4.23}$$

We find:

$$(b_\mu(x + a^{(\lambda)}) - b_\mu(x))\psi(x) = \frac{\pi}{2} \frac{n_{\mu\lambda}}{a_\lambda} \sigma_3\,. \tag{4.24}$$

---

5 Wilson loops winding around T^4 have to be invariant. See also footnote 4. The appropriate Wilson loop winding once around the torus in the ν^{th} direction is now $\mathrm{Tr}\left(P\exp\left(i\oint_{c_x} A_\mu dx_\mu\right)\Omega_\nu(x)\right)$

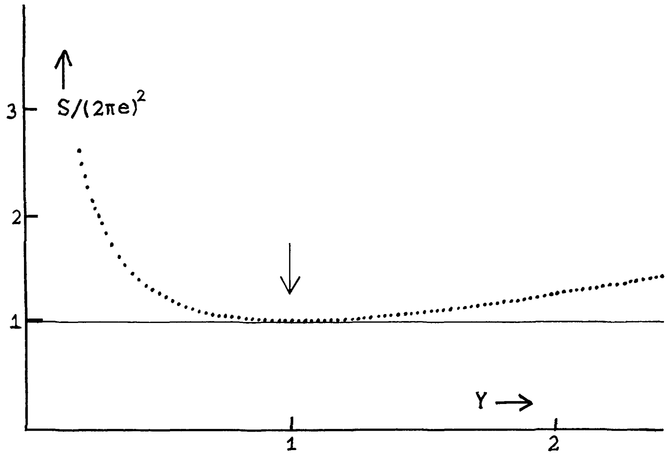

Fig. 1. The action S as a function of y. The dotted line is the abelian and the full line the antiselfdual solution

This is only possible if $\psi(x)$ is a multiple of σ_3, but then A_μ is necessarily of the abelian form [see (4.16)]

$$A_\mu = \left(-\frac{\pi}{2} F_{\mu\nu} x_\nu + \frac{\varphi_\mu^{(1)}}{a_\mu} \right) \sigma_3 \,. \tag{4.25}$$

Since we considered the case that this solution is not (anti)-selfdual, we conclude that the number of zero modes for M_A, with A (anti)-selfdual is $4(N-1)\,|\mathrm{Pf}(n)|$ and M_{gh} has no zero-modes.

There is a nice intuitive picture which explains the enlargement of the (anti)-selfdual solution manifold for (anti)-selfdual $F_{\mu\nu} = \frac{n_{\mu\nu}}{a_\mu a_\nu}$. For convenience fix $n_{13} = n_{24} = e$, $n_{12} = n_{14} = n_{23} = n_{24} = 0$ and put $y = a_1 a_3 / a_2 a_4$ which is variable. For an anti-selfdual solution we must have $y = 1$. The action of the abelian solution in (4.25) is:

$$S_a = 2\pi^2 e^2 (y + y^{-1}) \,. \tag{4.26}$$

This is depicted in Fig. 1. The unstable solution for $y \neq 1$, joins together with the anti-selfdual solution solution at $y = 1$. At this point the solution manifold is obviously enlarged. As we computed there are $4(N-1)$ extra zero-modes. In [18] one can find that the contribution to the path-integral, in so far as it depends on the coupling constant is

$$g^{-\sigma(F)} \exp\left(\frac{-4\pi^2 |\mathrm{Pf}(n)|}{g^2} \right), \tag{4.27}$$

where $\sigma(F) = h_1(F) - h_0(F) - (h_1(0) - h_0(0))$, with $h_1 = \dim \ker M_A$ and $h_0 = \dim \ker M_{gh}$. So $\sigma(F) = 4|\mathrm{Pf}(n)|$ if F is (anti)-selfdual and $\sigma(F) = 4|\mathrm{Pf}(n)| - 3$ if F is not. Thus tunneling through (anti)-selfdual solutions in the case that conditions (4.1) are not fulfilled is suppressed by a factor g^3.

5. Discussion

We considered on T^4 abelian solutions of the Yang-Mills equations of motion, satisfying abelian boundary conditions. In Appendix B we prove that this saturates all possibilities for SU(2) (up to a gauge), but for SU(N), $N>2$, t' Hooft's analysis [5] shows that abelian solutions exist which do not satisfy abelian boundary conditions.

Our computations suggest that the (anti)-selfdual solutions which contain an abelian sector dominate the path integral. Perhaps it is sufficient only to consider these situations. Furthermore in this case the valley in the action functional, describing the local minimum, is widest near the abelian solutions of the form $A_\mu = -\frac{\pi}{N} F_{\mu\nu} x_\nu T$. It is then feasible that one only needs to expand around these solutions. For this situation one can already check that the correct renormalization group behaviour is obtained, by explicitly using the results of Eq. (4.4). Also the quasi-classical expansions for M_A and M_{gh} given in [21] can be used.

However, since the "instantons"[6] on T^4 (after transforming to the $A_0=0$ gauge) do not in general represent tunneling between vacuum states, it is not guaranteed that the "instantons" we suspect to dominate the path integral interpolate between states with nonzero overlap with the vacuum states. In that case one is computing energy splits between excited states.

Alternatively one can try to find all euclidean solutions explicitly interpolating between Lüscher's vacuum configurations. This seems as difficult as constructing the general "instantons" on T^4. The subject is presently under investigation.

Finally we hope that our work is also of mathematical interest. It is suggested that θ-functions are the natural objects to construct the "instantons." Since θ-functions are algebraic objects on a torus, an algebraic construction, somewhat similar to S^4 (see [22] for a review), might be possible. Unlike S^4, we have on T^4 a "preferred" complex structure. One can even view our abelian solution in the holomorphic gauge $A_{z_i} = i\pi f_i \bar{z}_i$ as canonically associated with the metric on the U(1)-line bundle, admitting the θ-functions f as sections. The norm n is given by:

$$n(f(z)) = |f(z)| e^{-\frac{\pi}{2}H(z,z)}, \tag{5.1}$$

and the connection in the holomorphic gauge is then found by

$$iA_{z_i} = e^{\frac{\pi}{2}H(z,z)} \frac{\partial}{\partial z_i} e^{-\frac{\pi}{2}H(z,z)}. \tag{5.2}$$

Appendix A

We will discuss the necessary details to prove the following formula [remember that $L(H,\alpha)$ is the linear space of θ-functions of type (H,α), and not the line bundle determined by (H,α)],

$$\dim_{\mathbb{C}} L(H,\alpha) = |\mathrm{Pf}(n)|, \tag{A.1}$$

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6 Instantons between quotation marks, since we include twist

and to construct explicitly an orthonormal basis for $L(H,\alpha)$. For the time being we assume that H is nondegenerate. An obvious generalization of the intrinsic theta function f_0 (3.24) is:

$$f_t(z)=\sum_{q\in L}\alpha(q)\exp\left(\pi H(z+t,q-t)-\frac{\pi}{2}H(q,q)+\frac{\pi}{2}H(t,t)\right),\qquad E(t,q)\in\mathbb{Z}\quad\forall q\in L. \tag{A.2}$$

Explicitly the condition for t is:

$$n_{\mu\nu}t_\nu\in\mathbb{Z}. \tag{A.3}$$

Equivalently one can define f_t through:

$$f_t(z)=e^{-\frac{\pi}{2}H(t,t)-\pi H(z,t)}f_0(z+t). \tag{A.4}$$

Therefore t is only defined modulo L. Clearly, t satisfying (A.3) form a lattice L_n which contains L. If (A.1) is correct, then not all these θ-functions can be independent, since L_n/L has $\det(n)=\mathrm{Pf}(n)^2$ points. In the following we will show how to single out the appropriate set.

But first we will discuss the classical θ-functions which will yield a canonical orthonormal base for $L(H,\alpha)$. We remind the reader of the fact that n can be brought in the Frobenius standard form by an $SL(4,\mathbb{Z})$ transformation T [15, p. 71]:

$$n=T^t n^0 T,\qquad n^0=\begin{pmatrix} \multicolumn{2}{c}{\emptyset} & e_1 & 0\\ & & 0 & e_2\\ -e_1 & 0 & \multicolumn{2}{c}{\emptyset}\\ 0 & -e_2 & & \end{pmatrix} \tag{A.5}$$

$$e_1=\text{g.c.d.}(n_{\mu\nu}),\qquad e_2=\frac{-\mathrm{Pf}(n)}{e_1}\in e_1\mathbb{Z}.$$

Let the $\mathbb{Z}$-basis of the lattice L with respects to which n is of this form be $\xi^{(\mu)}$, i.e. $E(\xi^{(\mu)},\xi^{(\nu)})=n^0_{\mu\nu}$, or equivalently $\xi^{(\mu)}=T_{\mu\nu}\zeta^{(\nu)}$, and choose a $\mathbb{C}$-basis according to

$$z=\tilde z_1\frac{\xi^{(3)}}{e_1}+\tilde z_2\frac{\xi^{(4)}}{e_2}=U^t\tilde z. \tag{A.6}$$

Using $\mathrm{Im}\,H(z,w)=E(z,w)$ and $E(\xi^{(3)},\xi^{(4)})=0$, we find that $\tilde h=U^+hU$ is real and symmetric, so we can define the symmetric $\mathbb{C}$-bilinear form

$$S(z,w)=-\tilde z\tilde h\tilde w. \tag{A.7}$$

Next we introduce the mixed quadratic form,

$$Q(z,w)=H(z,w)+S(z,w), \tag{A.8}$$

and transform the θ-function f of type (H,α) to the θ-function θ of type (Q,α):

$$\theta(z)=\exp\left(\frac{\pi}{2}S(z,z)\right)f(z), \tag{A.9}$$

$$\begin{aligned}\theta(z+q)&=\tilde{u}_q(z)\theta(z); \qquad q\in L,\\ \tilde{u}_q(z)&=\exp\left(\frac{\pi}{2}Q(q,q)+\pi Q(z,q)\right)\alpha(q).\end{aligned} \tag{A.10}$$

It is obvious that the transformation between (H,α) and (Q,α) is $1-1$. It is introduced because of the property:

$$Q(z,\xi^{(3)})=Q(z,\xi^{(4)})=0, \tag{A.11}$$

or that $Q(z,w)$ vanishes for $\tilde{w}_i$ real. So for $q\in L$ of the form $p_1\xi^{(3)}+p_2\xi^{(4)}$, $\tilde{u}_q(z)=\alpha(q)$, and θ is almost periodic in the real direction. To work this out in more detail, we introduce the so-called characters m and l of α:

$$\alpha(q)=\exp(\pi iB(q,q))\beta(q), \tag{A.12}$$

where $\exp(\pi iB(q,q))$ is a bicharacter as in (2.19) but with respect to the Frobenius bases $(\xi^{(\mu)})$ of L; if we denote the components by $\tilde{q}_\mu(q=\tilde{q}_\mu\xi^{(\mu)})$,

$$B(q,q)=\sum_{i=1}^{2}e_i\tilde{q}_i\tilde{q}_{i+2}. \tag{A.13}$$

$\beta(q)$ is now obviously linear in q and this defines the characters of α:

$$\beta(q)=\exp\left(2\pi i\sum_{k=1}^{2}(m_k\tilde{q}_{k+2}-l_k\tilde{q}_k)\right). \tag{A.14}$$

For α given by (2.19) m and l can always be chosen 0 or $\frac{1}{2}$. With these definitions one easily verifies that the holomorphic function[7]:

$$\theta(z)\exp(-2\pi im\cdot e^{-1}\tilde{z}) \tag{A.15}$$

is periodic for $\tilde{z}$ real, with periods (e_1,e_2). Therefore θ has the following unique Fourier expansion:

$$\theta(z)=\sum_{r\in\mathbb{Z}^2}c(r)\exp(2\pi i(r+m)\cdot e^{-1}\tilde{z}). \tag{A.16}$$

We still have to satisfy (A.10) for $q=p_1\xi^{(1)}+p_2\xi^{(2)}$,

$$\theta(z+p)=\theta(z)\exp(-2\pi i(p\cdot(z+l)+\tfrac{1}{2}p\cdot\tau p)), \tag{A.17}$$

where we used the following identity:

$$Q(\xi^{(\mu)},\xi^{(\nu)})=Q(\xi^{(\nu)},\xi^{(\mu)})+2iE(\xi^{(\mu)},\xi^{(\nu)}). \tag{A.18}$$

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7 $e=\begin{pmatrix}e_1 & 0\\ 0 & e_2\end{pmatrix}$, $m\cdot\tilde{z}=m_1\tilde{z}_1+m_2\tilde{z}_2$

Using (A.11) this implies $Q(z,q)=-2ip\cdot z$. On the other hand $E(\xi^{(i)},\xi^{(j)})=0$ $(i,j=1,2)$ implies that

$$\tau_{ij}\equiv 2iQ(\xi^{(i)},\xi^{(j)}) \tag{A.19}$$

is symmetric. τ as defined above is called the period matrix, and one easily verifies that

$$\xi^{(i)}=\sum_{j=1}^{2}\tau_{ij}\frac{\xi^{(j+2)}}{e_j}, \tag{A.20}$$

and that $\mathrm{Im}\,\tau$ is positive definite,

$$\mathrm{Im}\,\tau=\tilde{h}^{-1}. \tag{A.21}$$

Substituting (A.16) in (A.17) gives a relation between the Fourier coefficients,

$$c(r+ep)=\exp(2\pi i(\tfrac{1}{2}p\cdot\tau p+(r+m)\cdot e^{-1}\tau p+p\cdot l))c(r),\quad p\in\mathbb{Z}^2 \tag{A.22}$$

and so we find:

$$\theta(z)=\sum_{0\leqq r_i<|e_i|}d(r)\theta_r(z), \tag{A.23}$$

$$\theta_r(z)=\sum_{p\in\mathbb{Z}^2}\exp(\pi i[p+e^{-1}(m+r)]\cdot\tau[p+e^{-1}(m+r)]+2\pi i(p+e^{-1}(m+r))\cdot(\tilde{z}+l)).$$

By construction we verified (A.1) since $|\mathrm{Pf}(n)|=|e_1e_2|$, but furthermore the θ_r form an orthogonal set of θ-functions each with the same length, where the inner product is defined through

$$\langle\theta_1,\theta_2\rangle=\int_{T^4}d_4x\,\theta_1(z)\overline{\theta_2(z)}\exp(-\pi\mathrm{Re}(Q(z,z))), \tag{A.24}$$

or if we write things in terms of the original U(1) sections,

$$\chi=\exp\left(-\frac{\pi}{2}Q(z,z)\right)\theta(z), \tag{A.25}$$

it is the standard inner product $\langle\chi_1,\chi_2\rangle=\int_{T^4}d_4x\,\chi_1(z)\overline{\chi_2(z)}$. For the proof of

$$\langle\theta_r,\theta_s\rangle=\delta_{rs}\langle\theta_0,\theta_0\rangle=\delta_{rs}\|\theta_0\|^2 \tag{A.26}$$

we refer the reader to the literature [15, p. 80]. The canonical basis for the zero-modes of M_n is therefore given by the orthonormal set:

$$\chi_r=\exp\left(-\frac{\pi}{2}Q(z,z)\right)\theta_r(z)/\|\theta_0\|,\quad r=(r_1,r_2),\quad r_i\in\mathbb{Z},\quad 0\leqq r_i<|e_i|, \tag{A.27}$$

which is still true if H is degenerate $(e_2=0)$, e_1 is called the reduced Pfaffian [15][8].

8 In (A.6) replace e_2 by 1, then following the analysis one finds (A.23) with all 2 comp. vectors replaced by their first component. So $\theta_r(z)$ is independent of $\tilde{z}_2$

There is one disadvantage concerning the above construction and that is its complexity. The intrinsic θ-functions (A.2) are on the other hand in general not orthogonal[9]:

$$\langle f_t, f_s\rangle = \int d_4x f_t(z)\overline{f_s(z)}e^{-\pi H(z,z)} = \exp(-\pi i E(t,s)) f_{t-s}(0)/f_1 f_2\,, \quad \text{(A.28)}$$

so they all have the same norm, but cannot all be orthogonal. Let us at least constitute a basis for $L(H,\alpha)$, using these intrinsic θ-functions. First we have

$$f_t(z) = e^{-\frac{\pi}{2}S(z,z)} \sum c_{tr}\theta_r(z)\,. \quad \text{(A.29)}$$

Using (A.4) we find

$$f_t(t) = e^{-\frac{\pi}{2}Q(t,t) - \pi Q(z,t) - \frac{\pi}{2}S(z,z)} \sum c_{or}\theta_r(z+t)\,. \quad \text{(A.30)}$$

A set of $|\mathrm{Pf}(n)|$ distinct $t \in L_n/L$, such that $Q(\cdot, t) = 0$ is given by

$$t = t_1\frac{\xi^{(3)}}{e_2} + t_2\frac{\xi^{(4)}}{e_2}; \quad 0 \leqq t_i < |e_i|\,, \quad t_i \in \mathbb{Z}\,. \quad \text{(A.31)}$$

With (A.23) this implies:

$$f_t(z) = e^{-\frac{\pi}{2}S(z,z)} \sum c_{or} e^{2\pi i(m+r)\cdot e^{-1}\tilde{t}}\theta_r(z)\,. \quad \text{(A.32)}$$

And so

$$c_{tr} = \exp(2\pi i(m+r)\cdot e^{-1}\tilde{t})c_{or}\,, \quad \text{(A.33)}$$

$f_t(z)$ form a basis for $L(H,\alpha)$ iff the square metric c_{tr} in (A.29) is nonsingular. Or

$$\det c = \exp\left(\pi i \sum_k m_k(e_k - 1)\right) \prod_r c_{or} \prod_{e_1 > k > l \geqq 0} (e^{2\pi i k/e_1} - e^{2\pi i l/e_1}) \prod_{|e_2| > k > l \geqq 0} (e^{2\pi i k/e_2} - e^{2\pi i l/e_2}) \neq 0\,. \quad \text{(A.34)}$$

So $c_{or} \neq 0$ $\forall r$ is necessary and sufficient. We leave this to the reader to verify [16, Sect. 4].

Note that $\langle f_t, f_t\rangle = \sum |c_{or}|^2 \|\theta_0\|^2$ independent of t, consistent with (A.28). Using (A.4) only, a somewhat simpler version of (A.28) can be established:

$$\langle f_t, f_s\rangle = e^{-\pi i E(t,s)}\langle f_{t-s}, f_0\rangle\,. \quad \text{(A.35)}$$

Also with (A.4) we have $f_{t-s}(0) = e^{-\frac{\pi}{2}H(t-s,t-s)} f_0(t-s)$. So knowledge of $f_0(t)$ enables us to find with the Gramm-Schmidt procedure an orthonormal basis, using (A.28).

We will end this appendix by mentioning a simple consequence of (A.28): All f orthogonal to f_0 have to vanish in $z=0$, and therefore on the whole of L.

9 (A.28) is found by explicitly working out the double sums, and using periodicity of the integrand

Appendix B

In this appendix we will answer two questions about $SU(N)/Z_N$ fiber bundles over T^4.

(i) Given a solution of the Yang-Mills equation ($D_\mu G_{\mu\nu}=0$) with constant curvature, is there a gauge in which the boundary conditions are of the abelian type?

(ii) For which values of the twist tensor $n_{\mu\nu}$ and Pontryagin-index P is there a gauge in which the boundary conditions are of the abelian type?

Let us first consider SU(2) and neglect the boundary conditions. In that case it was shown by Leutwyler [7] that all constant curvature solutions are of the abelian type up to a gauge

$$A_\mu = -\tfrac{1}{2}G_{\mu\nu}x_\nu ; \qquad G_{\mu\nu} = \pi F_{\mu\nu}\sigma_3 . \tag{B.1}$$

Now we impose the boundary conditions and write

$$\Omega_\nu(x) = \exp\left(\frac{\pi i}{2}\sum_\mu x_\mu F_{\mu\nu} a_\nu \sigma_3\right)\omega_\nu(x) . \tag{B.2}$$

From Eq. (2.3) one deduces

$$D_\mu \omega_\nu = 0 . \tag{B.3}$$

With a similar observation as in (4.18) we find

$$(D_\mu D_\lambda - D_\lambda D_\mu)\omega_\nu = i[G_{\mu\lambda}, \omega_\nu] = 0 . \tag{B.4}$$

So ω_ν commutes with σ_3. Combining with (B.3) this implies that ω_ν is a *constant* gauge function of the form:

$$\omega_\nu(x) = \exp(i\varphi_\nu \sigma_3) ; \qquad \varphi_\nu \text{ constant.} \tag{B.5}$$

Finally we transform ω_ν to the identity by the gauge transformation

$$\Omega(x) = \exp\left(-i\sum_\mu \frac{x_\mu \varphi_\mu}{a_\mu}\sigma_3\right) , \tag{B.6}$$

under which A_μ changes into

$$A_\mu = -\frac{\pi}{2}F_{\mu\nu}x_\nu \sigma_3 + \frac{\varphi_\mu}{a_\mu}\sigma_3 , \tag{B.7}$$

which is exactly the general abelian solution for SU(2) [see (4.14) and (4.16)]. Note that the cocycle condition forces $F_{\mu\nu}$ to be of the form $n_{\mu\nu}/a_\mu a_\nu$.

For $SU(N)$ ($N>2$) we assume the constant curvature solution to be abelian [for SU(2) this is automatically satisfied. Whether this is also true for $SU(N)$, $N>2$, is not relevant for the point we want to make],

$$[G_{\mu\nu}, G_{\lambda\sigma}] = 0 . \tag{B.8}$$

In a suitable gauge we have

$$A_\mu = -\tfrac{1}{2}G_{\mu\nu}x_\nu , \tag{B.9}$$

with boundary conditions

$$\Omega_\nu(x)=\exp\left(\frac{i}{2}\sum_\mu x_\mu G_{\mu\nu}a_\nu\right)\omega_\nu\,. \tag{B.10}$$

Again ω_ν is a constant gauge function satisfying

$$[\omega_\nu, G_{\mu\lambda}]=0\,. \tag{B.11}$$

Imposing the cocycle condition gives

$$\exp(ia_\mu G_{\mu\nu}a_\nu)\,[\omega_\nu,\omega_\mu]=\exp(2\pi i n_{\mu\nu}/N)\,. \tag{B.12}$$

There is a gauge transformation which brings ω_ν to the identity for all ν, if and only if $[\omega_\mu,\omega_\nu]=1$. The possibility to have $[\omega_\mu,\omega_\nu]\neq 1$ enabled 't Hooft to construct a constant curvature solution with Chern index $1/N$ [5]. So our first question can only be answered by yes for SU(2). Note that for $[\omega_\mu,\omega_\nu]=1$ the generalization following Eq. (2.27) is applicable with $A_\mu=A_\mu^0+\dfrac{1}{a_\mu}\mathrm{diag}(\varphi_\mu^{(1)}\ldots\varphi_\mu^{(N)})$; $\sum\varphi_\mu^{(i)}=0$ [compare (2.33)] as a general solution.

The second question will be considered for SU(2) only (see [12, Lemma 3.2], also for $N>2$). The abelian boundary conditions (4.23) uniquely fix C_2 to be $\frac{1}{2}\mathrm{Pf}(n)$. Therefore if $n=0 \bmod 2$ (no twist) C_2 is always even. There is no constant curvature solution with odd Chern index. For unit Chern index there seems even to be an obstruction for the existence of any solution satisfying the duality equations on T_4 [24]. For C_2 even, there certainly are (anti)-selfdual solutions however. Finally if the twist is non-zero ($n\neq 0 \bmod 2$) it is not hard to see that each value of C_2 compatible with the given twist can be reached. So to answer the second question: Only boundary conditions yielding $n_{\mu\nu}=2m_{\mu\nu}$ and $P_1=4(2k+1)$, with arbitrary m and k, are not gauge equivalent to abelian boundary conditions.

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