A simple construction of twist-eating solutions

Pierre van Baal
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11794
Bert van Geemen
Department of Mathematics, Budapestlaan 6, P. O. Box 80010, 3508 TA Utrecht, The Netherlands

(Received 7 May 1985; accepted for publication 30 September 1985)

A simple general construction of all solutions to the set of equations $[\Omega_\mu, \Omega_\nu] = \exp(2\pi i n_{\mu\nu}/N)I$, where $\Omega_\mu \in SU(N)$ or $U(N)$ and $\mu, \nu = 1, 2, \ldots, 2g$, is given.

I. REDUCTION TO A CANONICAL FORM

Twisted guage fields on the hypertorus, both in the continuum[1] and on the lattice,[2] posed the interesting mathematical problem of finding matrices Ω_μ in $SU(N)$ or $U(N)$ (called twist-eating solutions), such that

$$[\Omega_\mu, \Omega_\nu] = \Omega_\mu \Omega_\nu \Omega_\mu^{-1} \Omega_\nu^{-1} = \exp(2\pi i n_{\mu\nu}/N)I. \tag{1}$$

Here n is called the twist tensor; it is skew symmetric with integer entries mod N. The index μ runs from 1 up to $2g$ (the dimension of space-time; odd dimensions need not be considered separately). For details see Refs. 3 and 4, where the full solution of this problem for $g<2$ was found (see also Ref. 5).

By means of a Sl $(2g, \mathbf{Z})$ transformation X, we can always transform n to its standard[3] form n^s:

$$n^s = \begin{pmatrix} & \emptyset & e_1 & & \\ & & & \ddots & \\ & & & & e_g \\ -e_1 & & & & \\ & \ddots & & \emptyset & \\ & & -e_g & & \end{pmatrix}, \tag{2}$$

where $e_1|e_2|\cdots|e_g$ and $n = {}^tXn^sX$. (For integer p and q the symbol $p|q$ means that p divides q.) If $[\tilde\Omega_\mu, \tilde\Omega_\nu] = \exp(2\pi i n^s_{\mu\nu}/N)I$, then Eq. (1) is solved by

$$\Omega_\nu = \prod_\nu \tilde\Omega_\nu^{X_{\nu\mu}}. \tag{3}$$

The standard form n^s is not unique since we can add a multiple of N to each $n_{\mu\nu}$. However, transformation (3) is invertible[4]; the specific choice of n^s is therefore irrelevant. To be precise,

$$\tilde\Omega_\mu = Z_\mu \prod_\nu \Omega_\nu^{(X^{-1})_{\mu\nu}},$$

with Z_μ an element of the center of $SU(N)$, depending only on n and X.

Define

$$f_j = \gcd(e_j, N), \quad N_j = N_{g+j} = N/f_j, \quad j = 1, 2, \ldots, g. \tag{4}$$

(Greek indices will always run from 1 up to $2g$ and Latin indices from 1 up to g; gcd = greatest common divisor.) From the commutation relations it follows that

$$[\tilde\Omega_j^{N_j}, \tilde\Omega_{g+j}] = [\tilde\Omega_j, \tilde\Omega_{g+j}^{N_j}] = I.$$

Hence, the $\tilde\Omega_\mu^{N_\mu} \in SU(N)$ or $U(N)$ commute, so they can be simultaneously diagonalized. Let $A \in SU(N)$ be such that the

$$W_\mu = A\tilde\Omega_\mu^{N_\mu}A^{-1} \tag{5}$$

are diagonal matrices. As $[W_\mu, A\tilde\Omega_\mu A^{-1}] = I$ for all μ, ν we can choose diagonal matrices Λ_μ such that

$$\Lambda_\mu^{N_\mu} = W_\mu \text{ and } [\Lambda_\mu, A\Omega_\nu A^{-1}]. \tag{6}$$

If we define

$$\Omega'_\mu = \Lambda_\mu^{-1} A\tilde\Omega_\mu A^{-1}, \tag{7}$$

then the Ω'_μ satisfy

$$[\Omega'_\mu, \Omega'_\nu] = \exp(2\pi i n^s_{\mu\nu}/N)I, \quad (\Omega'_\mu)^{N_\mu} = I. \tag{8}$$

Next we will further simplify these commutation relations.

Recall that $\gcd(e_j/f_j, N_j) = 1$; hence there exist integers M_j such that

$$M_j(e_j/f_j) \equiv 1 \pmod{N_j}. \tag{9}$$

Define

$$U_j = (\Omega'_j)^{M_j}, \quad U_{g+j} = \Omega'_{g+j}. \tag{10}$$

This transformation can also be inverted: $\Omega'_j = U_j^{(e_j/f_j)}$, where we used that $(\Omega'_j)^{N_j} = I$. As $[U_j, U_{g+j}] = [(\Omega'_j)^{M_j}, \Omega'_{g+j}] = \exp(2\pi i e_j M_j/N)I$ and $e_jM_j/N = M_j(e_j/f_j)/N_j = N_j^{-1} \pmod{\mathbf{Z}}$, we see that the U_μ satisfy the commutation relations (1) with a twist tensor m in standard form:

$$m = \begin{pmatrix} & & f_1 & & \\ & \emptyset & & \ddots & \\ & & & & f_g \\ -f_1 & & & & \\ & \ddots & & \emptyset & \\ & & -f_g & & \end{pmatrix}. \tag{11}$$

(Note that $f_1|f_2|\cdots|f_g$ and moreover each f_j divides N.) In particular,

$$[U_j, U_{g+j}] = \exp(2\pi i N_j^{-1}), \quad (U_\mu)^{N_\mu} = I. \tag{12}$$

Hence to find all solutions to Eq. (1) it suffices to determine all solutions to Eq. (12).

II. THE GENERAL SOLUTION FOR THE CANONICAL FORM

Theorem: There exist matrices $U_\mu \in Gl(N)$ satisfying Eq. (12) if and only if $N_1N_2\cdots N_g$ divides N, where $N_i = N/f_i$.

Proof: Note that the subgroup K of $Gl(N)$ generated by

 0022-2488/86/020455-03$02.50

U_μ is finite. Moreover the U_j $(1 < j < g)$ generate an Abelian subgroup, in particular there is a basis of $\mathbb{C}^N$ consisting of simultaneous eigenvectors for all U_j $(1 \leqslant j < g)$. Let v be such a basis vector and assume $U_j v = \exp(2\pi i a_j/N_j)v$. Then $U_j\,(U_{g+j}v) = \exp(2\pi i(a_j+1)/N_j)U_{g+j}v$, hence the vectors Kv span a subspace V of dimension $N_1N_2\cdots N_g$. It is easy to see that K acts irreducibly on V. Proceeding with this method in the K-invariant complementary subspace of V we see that $\mathbb{C}^N$ is a direct sum of k K-invariant subspaces, each of dimension $N_1N_2\cdots N_g$. So $N = kN_1N_2\cdots N_g$.

To prove the converse, let V be a vector space of dimension $N_1N_2\cdots N_g$ with a basis $e(b_1, b_2,\dots, b_g)$, with $b_j \in \mathbb{Z}/N_j\mathbb{Z}$. Define linear maps $U'_\mu: V \rightarrow V$ by

$$\begin{aligned} U'_j e(b_1,b_2,\dots,b_g) &= \exp(2\pi i b_j/N_j)e(b_1,b_2,\dots,b_g), \\ U'_{g+j}e(b_1,\dots,b_j,\dots,b_g) &= e(b_1,\dots,b_j+1,\dots,b_g). \end{aligned} \tag{13}$$

It is easy to check that the U'_μ satisfy Eq. (12). Now assume $N = kN_1N_2\cdots N_g$, then $\mathbb{C}^N \cong V^k$ and define $U_\mu \in \mathrm{Gl}(N)$ by the block diagonal sum of k copies of U'_μ. Then obviously the U_μ also satisfy Eq. (12). ∎

We point out that the finite group K generated by the U_μ is a Heisenberg group. All irreducible representations were constructed in Ref. 6. Solutions of Eq. (12) form representations ρ of K, which, when restricted to the center $C(K)$ $(=\{\lambda I \,|\, \lambda^{N_1}=1\} \simeq Z_{N_1})$ of K, is given by $\rho(c) = c$, $\forall c \in C(K)$. This implies that each irreducible component of ρ has to be the unique so-called Schrödinger representation[6] [Eq. (13)]. Hence, ρ is unique up to a similarity transformation.

More directly, following closely the above proof of the theorem, it is easily seen that for $k=1$, $e(b_1, b_2, \dots, b_g)$ and $U_{g+j}^{(b_j-a_j)}v$ are to be identified. Similar statements for $k>1$ reproduce the block diagonal form, and two solutions to Eq. (12) have to be equivalent, i.e., $\exists A \in \mathrm{SU}(N)$, $U_\mu^{(2)} = AU_\mu^{(1)}A^{-1}$, $\forall \mu$. We will conclude this note with a few remarks.

The U_μ are unitary matrices. The explicit matrices for Eq. (13) are given by

$$\begin{aligned} U'_j &= \mathbb{1}_{N_1} \otimes\cdots\otimes Q_{N_j} \otimes\cdots\otimes \mathbb{1}_{N_g}, \\ U'_{g+j} &= \mathbb{1}_{N_1} \otimes\cdots\otimes P_{N_j} \otimes\cdots\otimes \mathbb{1}_{N_g}, \end{aligned} \tag{14a}$$

with

$$Q_n = \mathrm{diag}(1, e^{2\pi i/n}, \dots, e^{2\pi i(n-1)/n}),$$

$$P_n = \begin{pmatrix} 0 & 1 & & \emptyset \\ & & \ddots & \\ & 0 & & \\ \vdots & & & 1 \\ & & \ddots & \\ 1 & \cdots & & 0 \end{pmatrix}. \tag{14b}$$

This establishes the relation with the previous constructions.[3–5]

A solution Ω_μ to the original Eq. (1) is clearly specified by $A \in \mathrm{SU}(N)$ and Λ_μ, a diagonal unitary matrix [see Eqs. (5) and (6)], together with U'_μ [see Eqs. (13) and (14)]. Equation (6) implies that Λ_μ is a multiple of the identity in each block of U'_μ: $\Lambda_\mu = \mathrm{diag}(\lambda_\mu^{(1)}I, \dots, \lambda_\mu^{(k)}I)$, with I the $N_1N_2\cdots N_g$-dimensional identity matrix. Hence, the pair (A, Λ_μ) forms the group $G = \mathrm{SU}(N) \times \mathrm{U}(1)^k$. On the other hand, the uniqueness of solutions to Eq. (12) guarantees that for each $\{\Lambda_\mu\}$ satisfying $\Lambda_\mu^{N_\mu}$ for all μ, there exists an (in general not unique) $A \in \mathrm{SU}(N)$ such that for all μ,

$$\Lambda_\mu U'_\mu = AU'_\mu A^{-1}, \quad \Lambda_\mu^{N_\mu} = I. \tag{15}$$

[This can be explicitly verified for[3] Eq. (14).] Equation (15) specifies a subgroup H of G. The solutions to Eq. (1) are [for $\Omega_\mu \in \mathrm{U}(N)$] in 1–1 corresponding with G/H. These solutions are described by $2gk$ inequivalent continuous parameters [$2g(k-1)$ for $\Omega_\mu \in \mathrm{SU}(N)$]. A case of special interest is $k=1$ for $\Omega_\mu \in \mathrm{SU}(N)$, where the solution space for Eq. (1) modulo equivalence is discrete and isomorphic to $\Pi_{j=1}^g (Z_N/Z_{N_j})^2$, with $N^{2(g-1)}$ elements.

Suppose $N = k\,\Pi_{i=1}^g N_i$, define

$$m_i = -e_i/\gcd(e_i,N) = -(e_i/f_i). \tag{16}$$

Obviously both $n_{\mu\nu}$ and $N{\cdot}Pf(n/N)$ are multiples of k, since $e_i = -m_i k\,\Pi_{j\neq i}N_j$ and $N{\cdot}Pf(n/N) = -k\,\Pi_i m_i$. Consequently $N{\cdot}Pf(n/N) \in \mathbb{Z}$ is a necessary condition for existence of a solution to Eq. (1). Next observe that $\gcd(m_i, N_i) = 1$ and $N_g | N_{g-1} | \cdots | N_1$. Hence $\gcd(m_i, N_j) = 1$, for all $j \geqslant i$, so

$$\gcd(n_{\mu\nu}, N{\cdot}Pf(n/N), N) = k \gcd\left(\prod_{i=2}^g N_i, \prod_{i=2}^g m_i\right). \tag{17}$$

Given a solution, it is clearly unique up to a similarity transformation and Z_N factors if and only if $k=1$. Hence $\gcd(n_{\mu\nu}, N{\cdot}Pf(n/N), N) = 1$ is a sufficient condition for uniqueness. For $g=2$ it is also necessary, as can be seen from Eq. (17) and $\gcd(m_2, N_2) = 1$. Furthermore, in the case $g=2$, $N{\cdot}Pf(n/N) = -e_1e_2/N$. We can write $e_i = m_i f_i$, and $N = f_2 c$ with $\gcd(m_i, c) = 1$. Hence $N{\cdot}Pf(n/N) = -m_1m_2f_1/c \in \mathbb{Z}$ implies that f_1 is a multiple of c. So $N/N_1N_2 = f_1f_2/N = f_1/c \in \mathbb{Z}$. Consequently for $g=2$, $N{\cdot}Pf(n/N) \in \mathbb{Z}$ is also sufficient for existence of solutions to Eq. (1).

That the above criteria [i.e., $N{\cdot}Pf(n/N)$ is sufficient for existence and $\gcd(n_{\mu\nu}, N{\cdot}Pf(n/N), N) = 1$ is necessary for uniqueness] cannot be extended beyond $g=2$ can be seen from the following two examples constructed by Coste[7]: (i) $g=3$, $N = 2^2 3^6$, $e_1 = e_2 = 3^4$, and $e_3 = 2^4 3^4$ (hence $N_1 = N_2 = 2^2 3^2$ and $N_3 = 3^2$), so $N{\cdot}Pf(n/N) = e_1e_2e_3/N^2 = 1$ but $N_1N_2N_3 = 4N$ does not divide N, and no solution exists; and (ii) $g=3$, $N = 2^2 7^3$, $e_1 = e_2 = 2\cdot 3\cdot 7^2$, and $e_3 = 2^3\cdot 3\cdot 7^2$ (hence $N_1 = N_2 = 2\cdot 7$ and $N_3 = 7$), so $\gcd(n_{\mu\nu}, N{\cdot}Pf(n/N), N) = 2$, but $N_1N_2N_3 = N$ and the solution is unique.

Note added in proof: After completion of this work, we received a preprint by Lebedev and Polikarpov.[8] Their results coincide with those of Ref. 6 and this paper.

ACKNOWLEDGMENTS

We are grateful to Professor T. A. Springer for pointing out the connection with the Heisenberg group and Antoine Coste for the counterexamples to the $g=2$ criteria when extended to $g>2$.

B. v. G. is financially supported by the ZWO project on moduli and P. v. B. is supported in part by National Science Foundation Grant No. PHY-81-09110A-03.

[1]G. 't Hooft, Nucl. Phys. B **153**, 141 (1979); Acta Phys. Austriaca Suppl. **22**, 531 (1980).
[2]J. Groeneveld, J. Jurkiewicz, and C. P. Korthals-Altes, Physica Scripta **23**, 1022 (1981); A. Gonzales-Arroyo and M. Okawa, Phys. Rev. D **27**, 2397 (1983).
[3]P. van Baal, Commun. Math. Phys. **92**, 1 (1983).
[4]P. van Baal, "Twisted boundary conditions: A non-perturbative probe for pure non-Abelian gauge theories," thesis Utrecht, July 1984.
[5]J. Ambjørn and H. Flyvbjerg, Phys. Lett. B **97**, 241 (1980); G. 't Hooft, Commun. Math. Phys. **81**, 267 (1981); P. van Baal, Commun. Math. Phys. **85**, 529 (1982); Y. Brihaye, Phys. Lett. B **122**, 154 (1983); Y. Brihaye, G. Maiella, and P. Rossi, Nucl. Phys. B **222**, 309 (1983).
[6]B. van Geemen and P. van Baal, "The group theory of twist-eating solutions," Stony Brook preprint ITP-SB-84-86, to appear in Proc. K. Ned. Akad. Wet, Ser. B.
[7]A. Coste (private communication).
[8]D. R. Lebedev and M. I. Polikarpov, "Extrema of the twisted Eguchi–Kawai action and the finite Heisenberg group," preprint ITEP-41, 1985.