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LOCAL COMPACTIFICATION AND BLACK HOLES IN $d = 11$ SUPERGRAVITY

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We consider solutions of $d = 11$ supergravity describing a product of a four-dimensional Schwarzschild geometry and a seven-sphere whose radius R does depend on r. Three cases are studied as follows.

(i) Vanishing photon fields: an exact solution with r-dependent $R(r)$ is presented which exhibits a horizon.

(ii) Freund-Rubin asymptotics: Numerical results show that in this case solutions exist without a horizon at a finite value of r, but with $R(0) = \infty$.

(iii) Englert asymptotics: a striking analogy with the 't Hooft-Polyakov monopole is found which suggests the existence of a completely regular solution which interpolates between the Englert solution at $r = \infty$ and the Freund-Rubin solution at $r = 0$.

Most applications of Kaluza-Klein ideas to supergravity [1] have dealt with what we might call global (or rigid) compactification of space-time: above each point in four-dimensional space-time M_4 one has the same seven-dimensional compact manifold M_7, for example the round seven-sphere S_7 [2], the squashed seven-sphere [3], the $SU_3 \times SU_2 \times U_1$ spaces [4] or the $SU_2 \times SU_2 \times SU_2$ spaces [5]. In this article we will consider what we shall call local compactification [6–8]; we shall assume that the properties of M_7 may depend on M_4. As a model we will consider a black hole at the spatial origin of $d = 4$ Minkowski space-time, and consider above each point a round seven-sphere whose radius R may depend on its distance r from the origin. By allowing the three-index photon field also to depend on r, the four and seven dimensional spaces become interacting via this matter field. Another possibility for such an interaction which we will not pursue below would be to allow for off-diagonal terms in the $d = 11$ metric [7]. An interesting aspect of our model to which we intend to return in the future is that the effective four-dimensional gravitational coupling constant becomes r dependent, a phenomenon already encountered in another model by Cho and Freund [8] (see also Chodos and Detweiller [6]). This has led us to speculations concerning a running gravitational coupling constant and a

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relation between the Planck and GUT masses. Cosmological (i.e. time-dependent) solutions of $d = 11$ supergravity without torsion have been considered by Freund [6].

The reason we became interested in the model of a black hole $\otimes S_7(r)$ is a series of striking analogies with the 't Hooft-Polyakov monopole [9], as we shall discuss in more detail below. Whereas the latter can be viewed as a regular solution which interpolates between two particular solutions of the Yang-Mills-Higgs model, namely at $r = \infty$ and at $r = 0$, we will be looking for a regular solution of the field equations of $d = 11$ supergravity which interpolates between the Englert-Schwarzschild (ES) solution [10] at $r = \infty$ and a Freund-Rubin (FR) solution [1] at $r = 0$. In order that the interpolating solution be regular at $r = 0$, the energy density at the origin must be smooth and not contain the point mass which yields the $2m/r$ term in the usual $d = 4$ Schwarzschild solution. In particular we are interested in solutions whose regularity at the origin is suggested by the topology of the asymptotic ES solution.

The field equations of $d = 11$ supergravity read*

$$D_M F^{MNPQ} - \tfrac{1}{48} i\sqrt{2}\, \varepsilon^{NPQR_1 \dots R_4 S_1 \dots S_4} F_{R_1 \dots R_4} F_{S_1 \dots S_4} = 0\,, \tag{1}$$

$$R_{MN} = -96\left(F_{MPQR} F_N{}^{PQR} - \tfrac{1}{12} g_{MN} F^2_{PQRS} \right), \tag{2}$$

where we have set the gravitino field equal to zero and the one free parameter, the $d = 11$ gravitational coupling constant, equal to one.

We shall always assume that the metric be block diagonal in M_4 and M_7, but let us first in addition assume that $g_{\mu\nu}$ depends only on the coordinates x^μ of M_4 and $g_{\alpha\beta}$ only on the coordinates y^α of M_7. Adopting either the Freund-Rubin ansatz

$$F_{mnrs} = ib\varepsilon_{mnrs}\,, \qquad F_{abcd} = 0\,, \tag{3}$$

or the Englert ansatz

$$F_{mnrs} = ib\varepsilon_{mnrs}\,, \qquad F_{abcd} = cR^{-4} \bar{\eta} \Gamma_{abcd} \eta\,, \tag{4}$$

where $D_\alpha \eta = ia\Gamma_\alpha \eta$ and a, b, c are constants, then the Maxwell equation in (1) is satisfied if

$$c(a - 3\sqrt{2}\, b) = 0\,, \tag{5}$$

while the Einstein equation reduces to

$$R_{mn} = 192 g_{mn}(2b^2 + 7c^2 R^{-8})\,, \qquad R_{ab} = -192 g_{ab}(b^2 + 5c^2 R^{-8})\,. \tag{6}$$

* The photon curl has strength one, and $R_{MN} = \partial_N \Gamma^\Lambda_{M\Lambda} - \partial_\Lambda \Gamma^\Lambda_{MN} + \cdots$. The Dirac matrices satisfy $\{\Gamma_M, \Gamma_N\} = 2\delta_{MN}$ and $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$ where $M = 1,11$ and $a = 1,7$ and $m = 1,4$ are flat indices, while $\Lambda = 1,11$ and $\alpha = 1,7$ and $\mu = 1,4$ are curved indices. Note that $\varepsilon_{1\dots4} = \varepsilon^{1\dots4} = \varepsilon_{1\dots7} = \varepsilon^{1\dots7} = \varepsilon_{1\dots11} = \varepsilon^{1\dots11} = 1$, and $g^{(4)} = \det g_{\mu\nu}$, $g^{(7)} = \det g_{\alpha\beta}$.

For later convenience we have extracted the explicit factors of R in F_{abcd}. The consistency condition for the covariantly constant commuting spinor η reads

$$R_{ab} = -24a^2 g_{ab}, \tag{7}$$

and since on S_7 with radius R one has $R_{\alpha\beta} = -6g_{\alpha\beta}R^{-2}$, we see that $a = \pm\frac{1}{2}R^{-1}$. For both signs covariantly constant spinors exist on the round S_7; an explicit representation is given in [11]. In what follows we shall choose

$$a = \tfrac{1}{2}R^{-1}. \tag{8}$$

For non-vanishing c $(c \neq 0)$ $b = \frac{1}{12}\sqrt{2}\,R^{-1}$ according to (5) and $c = \pm\frac{1}{24}\sqrt{2}\,R^3$ according to (6) and (7). The case $c=0$ must be considered separately and in this case we find from (6) and (7) that $b^2 = \frac{1}{32}R^{-2}$.

It is clear from (6) that *any* Einstein metric on M_4 with proper cosmological constant yields a solution of the $d=11$ Einstein equations. In particular, if $g_{\mu\nu}$ is the Schwarzschild metric with cosmological constant, the $d=11$ solution which corresponds to the ansatz (3) will be called the FRS solution, while the ansatz (4) will be called the ES solution. In these solutions the radius R of S_7 is constant so that by moving in towards the black hole, there comes a point where the scalar curvatures of M_4 and M_7 become equal in absolute value. At this point one expects that one can no longer ignore interactions between M_4 and M_7, and one is led to consider an r-dependent radius of the seven-sphere.

To keep the maximal symmetry possible, we shall assume that (4) still holds but with $b=b(r)$ and $c=c(r)$, while the metric is the sum of a rotationally invariant static metric on M_4 and a maximally symmetric metric on M_7

$$-(\mathrm{d}s)^2 = -B(r)\,\mathrm{d}t^2 + A(r)\,\mathrm{d}r^2 + r^2(\mathrm{d}\Omega_2)^2 + R(r)^2(\mathrm{d}\Omega_7)^2,$$

$$(\mathrm{d}\Omega_n)^2 = \mathrm{d}\psi_n^2 + \sin^2(\psi_n)(\mathrm{d}\Omega_{n-1})^2. \tag{9}$$

Thus we have locally a direct product $M_4 \times M_7$ and the symmetry group is $R_1 \times SO_3 \times SO_8$. Covariantly constant spinors still exist although $g_{\alpha\beta}(y)$ has become r dependent, because the derivative in $D_\alpha\eta$ does not act on r. In fact, η is r independent but $a = \frac{1}{2}R^{-1}(r)$. It is advantageous for the computations ahead to introduce an r-independent sevenbein $\hat{e}^a_\alpha(y) = R^{-1}(r)e^a_\alpha(r,y)$. Let us define

$$A_{\alpha\beta\gamma} = ic(r)\bar{\eta}\hat{\Gamma}_{\alpha\beta\gamma}\eta = ic(r)R^{-3}(r)\bar{\eta}\Gamma_{\alpha\beta\gamma}\eta. \tag{10}$$

Then the non-vanishing components of F_{MNPQ} are in addition to (4)

$$F_{r\alpha\beta\gamma} = \tfrac{1}{4}\partial_r A_{\alpha\beta\gamma} = \tfrac{1}{4}ic'(r)\bar{\eta}\hat{\Gamma}_{\alpha\beta\gamma}\eta, \tag{11}$$

and we can proceed to solve the Maxwell equation (1).

The equation for $D_\Lambda F^{\Lambda\nu\rho\sigma}$ is no longer automatically satisfied but using that

$$\Gamma^{\alpha}_{\alpha\mu} = \tfrac{1}{2}\partial_r \ln g^{(7)} = 7R'R^{-1}, \qquad \text{if } \mu = r, \tag{12}$$

one readily finds the following integrable result

$$(bR^7)' = -21\sqrt{2}\,(c^2)'. \tag{13}$$

(The factor 21 is due to $\bar{\eta}\hat{\Gamma}_{\alpha\beta\gamma}\eta\bar{\eta}\hat{\Gamma}^{\alpha\beta\gamma}\eta = -42$.) Thus flux is conserved on S_7 when c is constant, for example in the Freund-Rubin case. (If one were to take the notion of the flux seriously, one might imagine $S_7(r)$ embedded into an eight (!) dimensional manifold; once again a hint at a theory of $d = 12$ supergravity [12].) Note that only for $c = 0$ the abelian "photons" A_{MNP} become non-self-interacting.

The $D_\Lambda F^{\Lambda\nu\rho\alpha}$ and $D_\Lambda F^{\Lambda\nu\alpha\beta}$ equations are still automatically satisfied, but the equation for $D_\Lambda F^{\Lambda\alpha\beta\gamma}$ modifies the result in (5) to

$$\left[c'(A^{-1}Br^4R^2)^{1/2}\right]' = 16cAR^{-2}(A^{-1}Br^4R^2)^{1/2}(1 - 6\sqrt{2}\,bR). \tag{14}$$

For constant but non-vanishing c, b still has the Englert value in (5), so that with (13) one concludes that in this case also b and R are constant. *Thus M_4 and M_7 only interact when c is not constant*, in which case all supersymmetries are broken [13], *or when $c = 0$*, in which case supersymmetry is preserved if $a(r) + 2\sqrt{2}\,b(r) = 0$, just as in the case of rigid compactification.

Having solved the Maxwell equation in (1), we now turn to the Einstein equation in (2). One can either directly compute the Ricci tensors from (9) or first construct an action for this line element. The Ricci tensor comes out as follows

$$R_{rr} = \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{A'}{A}\left(\frac{1}{r} + \frac{7}{2}\frac{R'}{R}\right) + 7\frac{R''}{R},$$

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{B'}{A}\left(\frac{1}{r} + \frac{7}{2}\frac{R'}{R}\right),$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A} + \frac{2}{r} + 14\frac{R'}{R}\right),$$

$$R_{\alpha\beta} = \left[-6 + \frac{R'R}{2A}\left(\frac{B'}{B} - \frac{A'}{A} + \frac{4}{r} + 12\frac{R'}{R}\right) + \frac{R''R}{A}\right] g_{\alpha\beta}R^{-2}. \tag{15}$$

Of course, the Ricci tensor for $R_{\alpha\beta}$ is still proportional to $g_{\alpha\beta}$. As a check one may note that for constant R one finds the well-known results for the Schwarzschild metric, while the R' terms are symmetric in S_2 and S_7 and the R'' terms are easily checked by a direct computation. We have also verified the Bianchi identities.

The Einstein equation now becomes

$$R_{rr} = 24A\left[16b^2 + 56c^2R^{-8} - 7(c')^2A^{-1}R^{-6}\right],$$

$$R_{tt} = -24B\left[16b^2 + 56c^2R^{-8} + \tfrac{7}{2}(c')^2A^{-1}R^{-6}\right],$$

$$R_{\theta\theta} = 24r^2\left[16b^2 + 56c^2R^{-8} + \tfrac{7}{2}(c')^2A^{-1}R^{-6}\right],$$

$$R_{\alpha\beta} = -24g_{\alpha\beta}\left[8b^2 + 40c^2R^{-8} + (c')^2A^{-1}R^{-6}\right]. \tag{16}$$

For constant c they reduce to (6), but there are new terms, proportional to $(c')^2$ and due to $F_{r\alpha\beta\gamma}$ in (11). As a check one may verify that the Bianchi identities remain satisfied if (13) and (14) hold. We now proceed to reduce the set of equations in (16) to two coupled equations for $R(r)$ and $c(r)$.

Just as in the usual Schwarzschild case, $R_{rr} + AB^{-1}R_{tt}$ is a useful combination, but here it yields an equation for R'' instead of the result that AB is constant

$$7\frac{R''}{R} = \left(\frac{A'}{A} + \frac{B'}{B}\right)\left(\frac{1}{r} + \frac{7}{2}\frac{R'}{R}\right) + A(\lambda_t - \lambda_r). \tag{17}$$

We have defined $R_{MN} = -g_{MN}\lambda_N$. From $AB^{-1}R_{tt}$ one finds a first-order equation for $B'B^{-1}$

$$\left(\frac{B'}{B}\right)' = -2A\lambda_t - \frac{B'}{rB}\left(A + 1 - Ar^2\lambda_\theta\right). \tag{18}$$

From $R_{\theta\theta}$ one may eliminate $A'A^{-1} - B'B^{-1}$

$$\frac{A'}{A} - \frac{B'}{B} = \frac{2}{r}(1 - A) + 14\frac{R'}{R} + 2r\lambda_\theta A. \tag{19}$$

Finally, from $R_{\alpha\beta}$ one finds with (19) an expression which is linear in A^{-1} and from which we can solve for A in terms of R, b and c

$$A(r) = \frac{r\left[r(\ln R)'\right]' + 12(rc')^2\left[7r(\ln R)' + 2\right]R^{-6}}{r(\ln R)'\left(r^2\omega_\theta - 1\right) + r^2\left(6R^{-2} - \omega_\alpha\right)},$$

$$\omega_\theta = -192(2b^2 + 7c^2R^{-8}), \qquad \omega_\alpha = 192(b^2 + 5c^2R^{-8}). \tag{20}$$

From the Bianchi identities we know that of the four equations (17)–(20) only three are independent. Thus we shall only keep (17), (19) and (20). Substituting into (17) the result for $B'B^{-1}$ as found in (19) and replacing everywhere A by the result in

(20) yields our final equation for $R(r)$

$$2r(\ln A)' = R^{-8}\left[r^2 R(R^7)'\right]'\left[1+\tfrac{1}{2}r(\ln R^7)'\right]^{-1}$$
$$+2-2A\left(1-r^2\omega_\theta\right)+84(rR^{-3}c')^2\left[1-r(\ln R^7)'\right]\left[1+\tfrac{1}{2}r(\ln R^7)'\right]^{-1}. \tag{21}$$

This is clearly a third-order differential equation for $R(r)$. To obtain our final equation for $c(r)$, we go back to (14) and eliminate $(A^{-1/2}B^{1/2})'A^{1/2}B^{-1/2} = \frac{1}{2}(B'B^{-1}-A'A^{-1})$ by means of (19). This yields a second-order equation for $c(r)$

$$c''+r^{-1}c'\left[A+1-6rR'R^{-1}-r^2A\omega_\theta+84(rc')^2R^{-6}\right]+16cAR^{-2}(6\sqrt{2}\,bR-1)=0. \tag{22}$$

Thus $R(r)$ and $c(r)$ are given by (21) and (22), $A(r)$ by (20), $b(r)$ by (13) and $B(r)$ by (19). When we demand R to tend to a constant $R(\infty)$ at $r=\infty$, also b and c have to become asymptotically constant when A does not tend to zero, see (13) and (14). When A tends to zero an asymptotic expansion is needed, see below. Using the invariance of all equations under $r\to\mu r$, $R\to\mu R$, $c\to\mu^3 c$, $b\to\mu^{-1}b$, we will put $R(\infty)=1$. Let us now consider first the case $b=c=0$, then the case $c=0$ but $b\neq 0$, and finally the general case.

Case (i). Vanishing photon fields, $b=c=0$. From (20) we find

$$A(r)=\left[r(\ln R)'\right]'\left[-r(\ln R)'+6r^2R^{-2}\right]^{-1}, \tag{23}$$

while (21) simplifies to

$$2rA'A^{-1}=2(1-A)+R^{-8}\left[r^2R(R^7)'\right]'\left[1+\tfrac{1}{2}r(\ln R^7)'\right]^{-1}. \tag{24}$$

Putting $R(r)=\alpha r$, (23) yields $\alpha^2=6$, and (24) becomes $rA'A^{-1}=8-A$. Thus we have the following exact solution

$$R(r)=\sqrt{6}\,r,\qquad A=8(1-2mr^{-8})^{-1},\qquad AB=1. \tag{25}$$

This solution has a horizon with topology $S_2\times S_7$ but it does not approach flat space-time at $r=\infty$ because A does not tend to unity. Note that the $d=11$ Schwarzschild solution for $(ds)^2=-B(r)dt^2+A(r)dr^2+r^2(d\Omega_9)^2$ is given by $A=(1-2mr^{-8})^{-1}$ with $AB=1$.

Case (ii). Vanishing $A_{\alpha\beta\gamma}$, $c=0$ but $b\neq 0$. In this case $\lambda_r=\lambda_t=\lambda_\theta=\omega_\theta$ in (20) and (17) yields

$$7\frac{R''}{R}=\left(\frac{A'}{A}+\frac{B'}{B}\right)\left(\frac{1}{r}+\frac{7}{2}\frac{R'}{R}\right). \tag{26}$$

With $R(\infty)=1$ and $bR^7=$ constant according to (13) and $b(\infty)^2=\frac{1}{32}R(\infty)^{-2}$ (valid as long as all terms in (15) vanish w.r.t. the term -6) we find $b^2=\frac{1}{32}R^{-14}$. Substituting this result into (19) yields an expression for $B'B^{-1}$ which, when substituted into (26), yields

$$2A'A^{-1}=7R''R^{-1}\left[r^{-1}+\tfrac{7}{2}R'R^{-1}\right]^{-1}+2r^{-1}(1-A)+14R'R^{-1}-24rAR^{-14}. \tag{27}$$

This is an equation for R containing only R because also A is only a function of R according to (20)

$$A(r)=r\left[r(\ln R)'\right]'\left[-r(\ln R)'\{1+12r^2R^{-14}\}+6r^2(R^{-2}-R^{-14})\right]^{-1}. \tag{28}$$

One may verify that asymptotically

$$\begin{aligned}
R&=1+qr^{-6}\left(1-\tfrac{15}{44}r^{-2}+\tfrac{1}{2}mr^{-3}+\cdots\right),\\
A&=(1-2mr^{-1}+4r^2)^{-1}\left(1-49qr^{-6}\{1+\mathcal{O}(r^{-2})\}\right),\\
B&=(1-2mr^{-1}+4r^2)\left(1-\tfrac{7}{4}qr^{-8}\left(1-\frac{3m}{r}\right)\{1+\mathcal{O}(r^{-2})\}\right),
\end{aligned} \tag{29}$$

which a posteriori justifies that $b(\infty)^2=\frac{1}{32}R(\infty)^2$. Except for $R(\infty)$ which we have put equal to one, there are two free parameters: the mass m and the "charge" q. To see what happens at $r=0$, we have made a numerical analysis of (27) for various values of m and q. The results in the figure show that for given m one can choose q large enough that there is no horizon, whereas at $q=0$ (and presumably small enough q) there is a horizon.

For positive q there is a singularity at $r=0$ and $R(0)=\infty$. Compared to the Reissner-Nordstrøm solution of ordinary gravity [14] our solution has a singularity of a new type [18].

The large cosmological constant in $d=4$ is an unwelcome property of all interesting Kaluza-Klein models. One can add by hand a cosmological constant to the $d=11$ action, which then breaks local supersymmetry explicitly [15], and yields the following Einstein equations

$$R_{MN}=-g_{MN}(\lambda_N+\Lambda). \tag{30}$$

The equations for non-vanishing Λ are simply obtained by replacing λ_M by $\lambda_M+\Lambda$ and ω_M by $\omega_M+\Lambda$. In the present case with $c=0$ one can fine-tune Λ such that the $d=4$ cosmological constant vanishes, namely by choosing $\Lambda=4R(\infty)^{-2}$ and $b(\infty)^2=\frac{1}{96}R(\infty)^{-2}$. In this case R approaches $R(\infty)$ exponentially fast. It is interesting to note that one cannot make the $d = 4$ cosmological constant zero by this mechanism if one has Englert asymptotics with $c(\infty) \neq 0$.

Case (iii). Englert asymptotics. *We shall interpret $c(r)$ as the Higgs field* whose vacuum expectation value $c(\infty)$ breaks the isotropy group $H = SO(7)$ of S_7 down to G_2. The SO(8) symmetry of S_7 is then broken down to $J = SO(7)$ [16]. (SO(7) is the subgroup of SO(8) which keeps a given spinor fixed, but this SO(7) is a different subgroup than the isotropy SO(7) which keeps the vector y^α fixed but under which no spinor is kept fixed.) Here an interesting analogy with the 't Hooft-Polyakov monopole comes to mind. In the Yang-Mills-Higgs model we consider the following three solutions, all with $A_0^a = 0$ for simplicity.

(i) The Wu-Yang solution [17] with $\phi^a = 0$, $A_i^a = \varepsilon_{aib} r^b r^{-2}$. This solution is gauge equivalent to a Dirac U(1) monopole but by the embedding into SO(3) the string singularity is avoided.

(ii) The point-singular 't Hooft-Polyakov monopole [9] with $\phi^a = c r^a r^{-1}$, $A_i^a =$ as before. Here one has added to the Wu-Yang monopole a scalar field with *constant* length c which is also covariantly constant

$$D_i \phi^a = \partial_i \phi^a + A_i^b (T^b)^{ac} \phi^c = 0, \qquad (T^b)^{ac} = \varepsilon^{abc}. \tag{31}$$

To make the analogy even stronger, note that one may write $\phi^a = c\eta^\dagger \tau^a \eta$ where η is

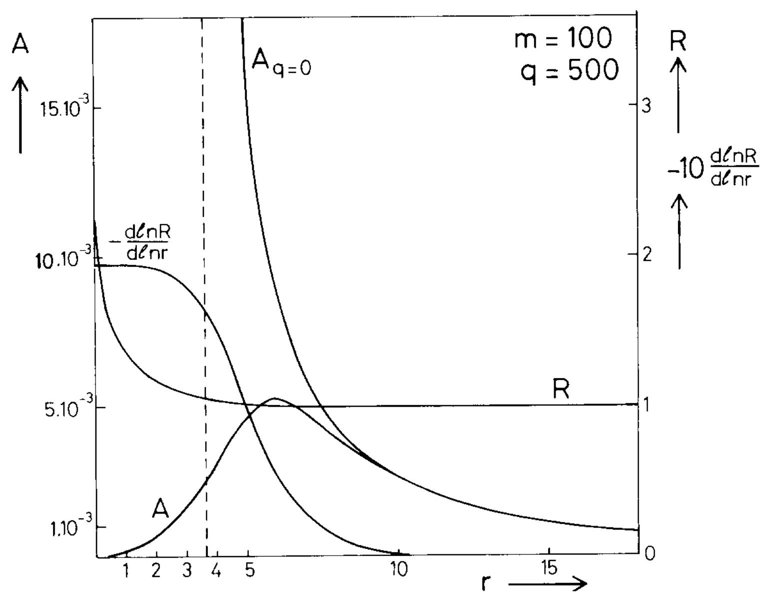

Fig. 1. Solution for $c = 0$, which approaches the Freund-Rubin-Schwarzschild solution asymptotically. The mass and charge parameters are equal $m = 100$ and $q = 500$. The solution exhibits a singularity at $r = 0$.

an SU_2 spinor

$$\eta = g^{-1}\eta_0, \qquad g^{-1} = \begin{pmatrix} \cos(\frac{1}{2}\theta) & -\sin(\frac{1}{2}\theta)e^{-i\varphi} \\ \sin(\frac{1}{2}\theta)e^{i\varphi} & \cos(\frac{1}{2}\theta) \end{pmatrix}, \qquad \eta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (32)$$

Since $d\eta = -\sigma\eta$ where $\sigma = g^{-1}dg$ satisfies $d\sigma + \sigma \wedge \sigma = 0$, η is SU_2 covariantly constant, just as η in supergravity is SO(8) (not merely SO(7)) covariantly constant.

(iii) The regular 't Hooft-Polyakov monopole [9] with $\phi^a = c(r)r^a r^{-1}$ and $A_i^a = w(r)\varepsilon_{iab}r^b r^{-2}$. For large r this solution approaches (ii) while for small r both $c(r)$ and $w(r)$ tend to zero. Let us now list the analogies with supergravity.

The FRS solution ($c=0$) is the analogue of the Wu-Yang solution. The ES solution ($c=$ constant) is the analogue of the point-singular monopole. We envisage the analogue of the regular monopole solution to be a solution of $d=11$ supergravity with $c(\infty)$ and $R(\infty)$ finite and non-vanishing but $c(0)=0$. If $R(0)$ and $b(0)$ would also be finite and non-vanishing one would have at $r=0$ a Freund-Rubin solution, but at $r=\infty$ an Englert solution.

It is well-known that the existence of the 't Hooft-Polyakov monopole is intimately linked to the topology of the Higgs field $\phi^a(\infty)$, which breaks G = SO(3) down to J = SO(2). The Higgs field maps a point (θ, φ) of S_r^2 at $r=\infty$ into a coset $G/J = SO(3)/SO(2) = S_\phi^2$. A similar situation is encountered for the Englert solution where $F_{\alpha\beta\gamma\delta}$ (or η) depends on the coordinates of S_r^7 and provides a mapping from S_r into $G/J = SO(8)/SO(7) = S_F^7$. Thus the topologically distinct (homotopy) classes are labelled by the winding number of the mapping $S_r^7 \to S_F^7$, i.e., by the elements of

$$\Pi_7(G/J) = \Pi_7(S_F^7) = Z. \quad (33)$$

Let us emphasize again that S_R^7 and S_F^7 are distinct seven spheres as they are related to two inequivalent SO_7 subgroups of SO_8. On the other hand, since both are coset spaces of the same SO_8 the winding number will be fixed. This is also suggested by the fact that the covariantly constant spinor used by Englert is in fact a unique solution up to the sign of a (which could flip the sign of the winding number) and some global SO_8 rotation (which does not affect the winding number). This is different from the monopole case where the boundary condition on the Higgs field can generate any winding number. It is however very similar to the monopole case if in addition one would insist on spherical symmetry under the mixed angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{T}$, in which case one only has the solutions with winding number ± 1.

It is easy to verify that the c equation (18) on a fixed gravitational background (say the ES solution) has no soliton type solution which smoothly interpolates between $c(\infty) = c_E$ and $c(0) = 0$. A non-trivial behaviour of R, A and B is therefore necessary, just like in the monopole case where the Higgs field has only the trivial solution if one fixes the gauge fields to be the Wu-Yang solution. Also there both the Higgs and the gauge fields are essential for obtaining a regular soliton type solution.

To really prove the existence of these regular solutions we have to solve our set of two coupled equations. In the monopole case the existence is a simple consequence of the positivity of the hamiltonian. Due to the gravitational interactions the hamiltonian is not bounded from below in our case and we look for a solution by integrating the equations with the boundary conditions $R \to 1$, $c^{-2} \to 288$ and $b \to \frac{1}{12}\sqrt{2}$ for $r \to \infty$ (the sign of c is irrelevant). The asymptotics are given by

$$R = 1 + q_1 f_1(r) r^{-b} + q_2 f_2(r) r^{-n} + \mathcal{O}(r^{-n-4}),$$
$$c = \tfrac{1}{24}\sqrt{2}\left(1 + 2q_1 f_1(r) r^{-b} - 12 q_2 f_2(r) r^{-n} + \mathcal{O}(r^{-n-4})\right), \tag{34}$$

where n satisfies $n^2 - 3n - 6 = 0$ and is given by $n = \frac{1}{2}(3 + \sqrt{33})$ and

$$f_1(r) = 1 - \tfrac{9}{22} r^{-2} + \tfrac{3}{5} m r^{-3},$$
$$f_2(r) = 1 - \tfrac{3}{10}(n+3)(2n-1)^{-1} r^{-2} + \frac{3m}{10}(n+2) n^{-1} r^{-3},$$
$$b = \left(\tfrac{5}{32} - 21c^2\right)\sqrt{2}\, R^{-7}. \tag{35}$$

R and c satisfy a third and a second order differential equation so we expect five integration constants. They are $R(\infty) = 1$, $c(\infty) = \frac{1}{24}\sqrt{2}$ and the free parameters m, q_1 and q_2 (respectively the mass and two "charges"). Furthermore we find

$$A = \left(1 + \tfrac{10}{3} r^2 - \frac{2m}{r}\right)^{-1}\left[1 + 49 q_1 r^{-6} + 7(n+1) q_2 r^{-n} + \mathcal{O}(r^{-n-2})\right],$$
$$B = \left(1 + \tfrac{10}{3} r^2 - \frac{2m}{r}\right)\left[1 - \tfrac{21}{10} r^{-2}\left(q_1 r^{-6} + q_2 r^{-n}\right)\left(1 - \frac{3m}{r}\right) + \mathcal{O}(r^{-n-4})\right]. \tag{36}$$

The main problem for finding a regular solution is that a 3-parameter search in m, q_1 and q_2 is needed. We leave a further numerical study of the complicated system of equations for R and c for a future publication.

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