Nuclear Physics B413 (1994) 535–552
North-Holland

NUCLEAR
PHYSICS B

Instantons from over-improved cooling

Margarita García Pérez [a], Antonio González-Arroyo [b], Jeroen Snippe [a] and Pierre van Baal [a]

[a] *Instituut-Lorentz for Theoretical Physics, University of Leiden, PO Box 9506, NL-2300 RA Leiden, The Netherlands*
[b] *Departamento de Física Teórica C-XI, Universidad Autónoma de Madrid, 28049 Madrid, Spain*

Received 20 September 1993
Accepted for publication 18 October 1993

Lattice artefacts are used, through modified lattice actions, as a tool to find the largest instantons in a toroidal geometry $[0,L]^3 \times [0,T]$ for $T \to \infty$. It is conjectured that the largest instanton is associated with tunnelling through a sphaleron. Existence of instantons with at least eight parameters can be proven with the help of twisted boundary conditions in the time direction. Numerical results for SU(2) gauge theory obtained by cooling are presented to demonstrate the viability of the method.

1. Introduction

Since the time of the discovery of instantons [1] in non-abelian gauge theories, as vacuum to vacuum quantum mechanical tunnelling events [2], their role in strongly interacting theories has been controversial, both in the continuum [3] and in the lattice formulation [4]. For the continuum this has been mainly due to applying semiclassical techniques, which cannot be justified at strong coupling. In the lattice formulation the main problems were the instantons localised at the scale of the lattice cut-off for which topological charge cannot be defined unambiguously [5], and which have actions considerably lower than the continuum action of $8\pi^2$. Strictly speaking, there are no locally stable solutions on a lattice using the standard Wilson action [6], because this lattice action decreases when the instanton becomes more localized [7], as we will demonstrate also from analytic considerations. On a trial and error basis, different (improved) lattice actions were considered, some of them indeed giving rise to stable lattice solutions [8]. This paper will provide the proper framework to understand the stability.

It is not too difficult to understand the reason of the instability. At finite lattice sizes the lattice action deviates from the continuum and this deviation is larger for stronger fields. For the Wilson action, as we will show, the lattice artefacts make the action decrease as compared to the continuum. In the continuum, instantons have a scale (or size) parameter ρ, on which the action does

SSDI 0550-3213(93)E531-4

not depend. But the smaller ρ becomes, the larger the fields get, which makes the lattice action *decrease*. On dimensional grounds one easily argues that (generically) $S_{\rm latt}(a,\rho) = 8\pi^2(1 + (a/\rho)^2 d_2 + {\rm O}(a/\rho)^4)$ for $\rho \gg a$, which will be demonstrated in more detail further on. For the Wilson action [6] $d_2 < 0$, explaining the instability. Hence, one simply modifies the action, such that $d_2 > 0$, in order to get stable solutions for the maximal value of ρ allowed by the volume $[0,L]^3$, which is kept finite. As we are interested in the classical solutions to the equations of motion, the modified action need not be of the type of an improved action [9], for which typically one wants to achieve $d_2 = 0$, as in that case (as we will show) the $(a/\rho)^4$ term might still destabilize the solution.

We deliberately want to keep $d_2 > 0$, which we will hence call over-improvement. The reason is, that our motivation for embarking on this project was to find the instantons with the largest scale ρ. This presumably will correspond to tunnelling over the lowest energy barrier, separating two classical vacua. The configuration that corresponds to the lowest barrier height is then conjectured to be a sphaleron (which exists due to the fact that we keep the volume finite). A sphaleron [10] is by definition a saddle point of the energy functional with precisely one unstable direction, which corresponds to the direction of tunnelling. In this way we use the instantons to map out the part of the energy functional relevant for the dynamical region where a semiclassical analysis of tunnelling amplitudes will break down. We refer to a pilot study [11] on $S^3 \times \mathbb{R}$ for readers interested in this issue, and for an explanation of the relevance of the geometry $T^3 \times \mathbb{R}$ which is studied in this paper. This geometry allows us to find the instantons using the lattice approximation. For simplicity we restrict ourselves to SU(2) pure gauge theories.

2. On the existence of continuum solutions

The geometry $T^3 \times \mathbb{R}$, in particular in a lattice formulation, can be seen as a limiting case of an asymmetric four torus $[0,L]^3 \times [0,T]$. The only known solutions have constant curvature [12] and hence cannot correspond to vacuum to vacuum tunnelling, furthermore their topological charge is at least 2. Actually, it can be proven rigorously [13], that for T finite, no regular charge-1 self-dual solutions can exist on a four-torus (we will illustrate this with our numerical results). As soon as we allow for twisted boundary conditions [14], existence of minimal non-trivial topological charge instanton solutions can be proven. One distinguishes two cases, depending on the properties of the twist tensor $n_{\mu\nu} \in \mathbb{Z}_2$.

When $\frac{1}{8}\epsilon_{\mu\nu\lambda\sigma} n_{\mu\nu} n_{\lambda\sigma} = 1 \bmod 2$, the topological charge is half-integer. The minimal action allowed by the topological bound is therefore $4\pi^2$, corresponding to topological charge 1/2. As twist is also well defined on the lattice [15], and in the above situation (called non-orthogonal twist) does not allow for zero-action configurations, these instantons cannot "fall through the lattice". Indeed, the index theorem predicts in this case four parameters ($8 \times$ topological charge), which have to correspond to the position parameters. The charge-1/2 instanton hence has fixed size and cannot shrink due to lattice artefacts. Impressively accurate results [16] were obtained for this case using the well known cooling method [7,17] to find a solution of the (lattice) equations of motion, whose smoothness and scaling with the lattice volume leaves no room to doubt it provides an accurate approximation to the continuum solution with action $4\pi^2$. In the continuum, existence of smooth non-trivial (but not necessarily self-dual) solutions was proven by Sedlacek [18], whereas theorem 3.2.1. of ref. [19] states that the moduli space of self-dual solutions is isomorphic with a four-torus.

When $\frac{1}{8}\epsilon_{\mu\nu\lambda\sigma}n_{\mu\nu}n_{\lambda\sigma} = 0 \bmod 2$, also called an orthogonal twist, there are "twist-eating" [15] configurations, i.e. configurations that have zero action *and* are compatible with twisted boundary conditions (see also ref. [21]). For SU(2), it is not too difficult to show that as long as $n_{\mu\nu} \neq 0 \bmod 2$ for some μ and ν, this twist eating configuration is unique [22], up to a global gauge transformation if a twist is introduced as in ref. [15,16] and multiplication with elements of the center of the gauge group. With twisted boundary conditions as originally defined by 't Hooft [14], such a global gauge transformation would even change the boundary conditions, and as an SO(3) bundle the twist-eating configuration is unique. (For SU(N) it can be proven [23] that out of the N^4 center elements that can multiply the twist, only N^2 give rise to gauge inequivalent configurations.) Under this condition it can be shown [19] that there are instanton solutions with 8 parameters (its moduli space, when dividing out the trivial translation parameters, is even related to a K3 surface [19,20]) using Taubes' [24] technique of glueing a localized instanton (with scale, position and global gauge parameters) to the "twist-eating" flat connection (i.e. zero action configuration). As the latter is *not* invariant under global gauge transformations, the global gauge parameters of the localized instantons are genuine parameters of the moduli space (see also ref. [25]).

The reason twisted boundary conditions are useful, is that at finite T there are no exact instantons on T^4 with periodic boundary conditions, but there are exact solutions for any non-trivial twist in the time direction. As $T \to \infty$ these solutions are also solutions on $T^3 \times \mathbb{R}$. This comes about as follows. Since at $T \to \infty$ the action can only stay finite if for $|t| \to \infty$ the energy density goes to zero, we deduce from a vanishing magnetic energy that up to a gauge

$$A_j(\boldsymbol{x}, t \to \pm\infty) = iC_j^{\pm}\sigma_3/2L, \tag{1}$$

where $C_j^{\pm} \in [0, 4\pi]$ ($A_0 = 0$) parametrizes the vacuum or toron valley [26], whose gauge invariant observables are best described by the Polyakov-line expectation values

$$P_i \equiv \tfrac{1}{2}\mathrm{Tr}\left(\mathrm{P}\exp\left(\int_0^L A_i(\boldsymbol{x},t)\mathrm{d}x_i\right)\right) = \cos\left(C_i/2\right)\,, \tag{2}$$

(for the proper definiton in the presence of twist, see ref. [16].) In the vacuum valley, P_i is space independent and the vanishing of the electric energy at $t \to \pm\infty$ also requires P_i (or $C_i^{\pm}$) to be asymptotically time independent. Instanton solutions on $\mathrm{T}^3 \times \mathbb{R}$ are hence characterized by the boundary conditions $C_i^{\pm}$ at $t \to \pm\infty$. It is these general instantons that are physically relevant. It is not clear if solutions exist with arbitrary boundary values. Approaching $T \to \infty$, by using periodic boundary conditions (which would impose $C_i^{+} = C_i^{-} \bmod 4\pi$ up to a periodic gauge transformation) does not allow us to prove existence. As long as T is finite there are no solutions [13] and the proof of non-existence breaks down as $T \to \infty$. On the other hand, with twist in the time direction, $n_{0i} = 1$, even at T finite there is in the continuum an eight parameter set of exact instanton solutions, which at $T \to \infty$ will correspond to $C_i^{+} = (2\pi - C_i^{-}) \bmod 4\pi$ (again up to a periodic gauge transformation). For localized instantons, asymptotically the field has to coincide with the unique flat connection, which fixes the possible values of $C_i^{\pm}$ to π, but at the other extreme, as the instanton in the spatial direction extends up to the "boundary" of the torus, the regions $t \to +\infty$ and $t \to -\infty$ no longer are connected, which will relax the fact that $P_i = 0$ ($C_i^{\pm} = \pi$). Although we have no proof, it is reasonable to assume that the eight parameters for the instantons close to the maximal size are described by ρ, the four position parameters and the three vacuum valley parameters C_i^{+} (or C_i^{-}). Note that for $P_i \to 0$ as $t \to \pm\infty$, the solution is both compatible with twisted *and* periodic boundary conditions *at infinite* T. In any case we have now learned that on $\mathrm{T}^3 \times \mathbb{R}$ (i.e. with free boundary conditions at $t \to \pm\infty$) there are, at least eight and at most eleven continuous parameters that describe the instanton solutions for vacuum to vacuum tunnelling.

3. The lattice actions and cooling

Let us start with discussing the standard Wilson action [6]

$$S = \sum_{x,\mu,\nu} \mathrm{Tr}\left(1 - {}_{\nu}\underset{x\ \ \mu}{\square}\right) = \sum_{x,\mu,\nu} \mathrm{Tr}(1 - U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x))\,, \tag{3}$$

where $U_\mu(x)$ are SU(2) group elements on the link that runs from x to $x+\hat{\mu}$, the latter being the unit vector in the μ direction. To derive the equations of motion, we observe that S depends on $U_\mu(x)$ through the expression

$$S(U_\mu(x)) = \mathrm{Tr}(1 - U_\mu(x)\tilde{U}_\mu^\dagger(x)) + \mathrm{Tr}(1 - U_\mu^\dagger(x)\tilde{U}_\mu(x))\,, \tag{4}$$

where

$$\tilde{U}_\mu(x) = \sum_{\nu\neq\mu} \left(\quad + \quad \right)$$
$$= \sum_{\nu\neq\mu} (U_\nu(x)U_\mu(x+\hat{\nu})U_\nu^\dagger(x+\hat{\mu}) + U_\nu^\dagger(x-\hat{\nu})U_\mu(x-\hat{\nu})$$
$$\times U_\nu(x+\hat{\mu}-\hat{\nu})), \tag{5}$$

which is independent of $U_\mu(x)$. Hence, $S(e^X U_\mu(x)) - S(U_\mu(x)) = O(X^2)$ for any Lie algebra element X, implies

$$\mathrm{Tr}[\sigma_i(U_\mu(x)\tilde{U}_\mu^\dagger(x) - \tilde{U}_\mu(x)U_\mu^\dagger(x))] = 0, \tag{6}$$

where σ_i are the Pauli matrices. This is easily seen to imply that $U_\mu(x)\tilde{U}_\mu^\dagger(x)$ is a multiple of the identity, and as $\tilde{U}_\mu$ is the sum of SU(2) matrices, it can be written as $\tilde{U}_\mu = a_0 + i\boldsymbol{a}\cdot\boldsymbol{\sigma}$, with $a_\mu \in \mathbb{R}^4$. If we define $\|\tilde{U}_\mu\| = (a_\mu^2)^{1/2}$, eq. (6) is seen to imply

$$U_\mu(x) = \pm\tilde{U}_\mu(x)/\|\tilde{U}_\mu(x)\|. \tag{7}$$

As we are only interested in stable solutions (i.e. local minima of the action), the plus sign in eq. (7) is the relevant one. The process of iteratively finding the solution to the equations of motion is called cooling [17], as in all cases it is devised such that the action is lowered after each iteration. The easiest is to simply choose $U'_\mu(x) = \tilde{U}_\mu(x)/\|\tilde{U}_\mu(x)\|$ since the fixed point of this iteration is clearly a solution to the equations of motion. An optimal way to sweep through the lattice is to divide for each μ the links $U_\mu(x)$ in two mutually exclusive checkerboard patterns Π_μ^i such that all links on a particular pattern Π_μ^i (i.e. for fixed i and μ) can be changed simultaneously, which is a well known trick to vectorize this procedure. At the cost of roughly a factor two in memory-use, vectorization is also achieved for the modified action we have considered so far for our numerical simulations:

$$S(\varepsilon) = \frac{4-\varepsilon}{3}\sum_{x,\mu,\nu}\mathrm{Tr}\left(1 - \quad\right) + \frac{\varepsilon-1}{48}\sum_{x,\mu,\nu}\mathrm{Tr}\left(1 - \quad\right). \tag{8}$$

The meaning of the parameter ε will become clear in the next section. For ease of our numerical studies we have not considered modified single plaquette actions (see also the next section for a discussion on the adjoint and Manton actions).

4. Lattice artefacts

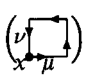
To calculate the effect of the discretization on the solutions of the equations of motion we first take a smooth continuum configuration (not necessarily a solution) $A_\mu(x)$. For definiteness we put $L = 1$, and N_s the number of lattice points in the spatial direction such that $a = 1/N_s$. We put this configuration on the lattice by defining:

$$U_\mu(x) = \mathrm{Pexp}\left(\int_0^a A_\mu(x + s\hat{\mu})\mathrm{d}s\right). \tag{9}$$

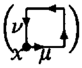
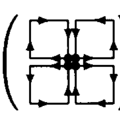
The value of the plaquette thus corresponds to parallel transport around a square and can easily be proven to be given by [27] ($D_\mu = \partial_\mu + A_\mu(x)$ the covariant derivative in the fundamental representation)

$\mathrm{Tr}\Big($  $\Big) = \mathrm{Tr}(\mathrm{e}^{aD_\mu(x)}\mathrm{e}^{aD_\nu(x)}\mathrm{e}^{-aD_\mu(x)}\mathrm{e}^{-aD_\nu(x)})\,.$ (10)

The proof simply amounts to observing that if $A_\mu(x) = A_\mu$, i.e. A_μ is space-time independent, then

$\mathrm{Tr}\Big($  $\Big) = \mathrm{Tr}(\mathrm{e}^{aA_\mu}\mathrm{e}^{aA_\nu}\mathrm{e}^{-aA_\mu}\mathrm{e}^{-aA_\nu})$

and eq. (10) is the only way to make this formula gauge invariant under arbitrary (i.e. x-dependent) gauge transformations. Using the Campbell–Baker–Hausdorff formula, eq. (10) can be expressed in terms of products of covariant derivatives $\mathcal{D}_\mu$ (in the adjoint representation) acting on the curvature $F_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, e.g. $\mathcal{D}_\mu F_{\mu\nu} = [D_\mu, [D_\mu, D_\nu]]$. As the action involves a sum over all x, μ and ν, things can be considerably simplified by computing, what we will call, the clover average

$\Big\langle \mathrm{Tr}\Big($  $\Big)\Big\rangle_{\text{clover}} = \frac{1}{4}\mathrm{Tr}\Big($  $\Big)$

$$\begin{aligned}
&= \frac{1}{4}\mathrm{Tr}\left[\mathrm{e}^{-aD_\mu}\mathrm{e}^{-aD_\nu}\mathrm{e}^{aD_\mu}\mathrm{e}^{aD_\nu} + \mathrm{e}^{-aD_\mu}\mathrm{e}^{aD_\nu}\mathrm{e}^{aD_\mu}\mathrm{e}^{-aD_\nu}\right.\\
&\qquad \left.+\mathrm{e}^{aD_\mu}\mathrm{e}^{-aD_\nu}\mathrm{e}^{-aD_\mu}\mathrm{e}^{aD_\nu} + \mathrm{e}^{aD_\mu}\mathrm{e}^{aD_\nu}\mathrm{e}^{-aD_\mu}\mathrm{e}^{-aD_\nu}\right]\\
&= \mathrm{Tr}\left[1 + \frac{a^4}{2}F^2_{\mu\nu}(x) - \frac{a^6}{24}\left((\mathcal{D}_\mu F_{\mu\nu}(x))^2 + (\mathcal{D}_\nu F_{\mu\nu}(x))^2\right)\right.\\
&\qquad + \frac{a^8}{24}\left\{F^4_{\mu\nu}(x) + \frac{1}{30}\left((\mathcal{D}^2_\mu F_{\mu\nu}(x))^2 + (\mathcal{D}^2_\nu F_{\mu\nu}(x))^2\right)\right.\\
&\qquad \left.\left. + \frac{1}{3}\mathcal{D}^2_\mu F_{\mu\nu}(x)\mathcal{D}^2_\nu F_{\mu\nu}(x) - \frac{1}{4}(\mathcal{D}_\mu\mathcal{D}_\nu F_{\mu\nu}(x))^2\right\}\right]\\
&\qquad + \mathrm{O}(a^{10}) + \text{total derivative terms},
\end{aligned} \tag{11}$$

for which the multiple Campbell–Baker–Hausdorff expansion of eq. (10) is required to $\mathrm{O}(a^6)$, obtained with the aid of the symbolic manipulation program FORM [28]. The clover average allows one to ignore many terms (all those odd in any of the indices) in evaluating the trace of the exponent.

Eq. (11) was also derived using the non-abelian Stokes formula [29] ($s_0 \equiv 1$)

$$\begin{aligned} U_{\mu\nu}(x) &\equiv U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x) \\ &= \mathrm{P}\exp\left[a^2\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\ \mathcal{F}_{\mu\nu}(x+as\hat{\mu}+at\hat{\nu})\right] \\ &\equiv 1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\int_0^{s_{i-1}}\mathrm{d}s_i\int_0^1 \mathrm{d}t_i\ a^2\mathcal{F}_{\mu\nu}(x+as_1\hat{\mu}+at_1\hat{\nu}) \\ &\qquad \times\ldots a^2\mathcal{F}_{\mu\nu}(x+as_n\hat{\mu}+at_n\hat{\nu})\,, \end{aligned} \tag{12}$$

where $\mathcal{F}_{\mu\nu}(y)$ equals $F_{\mu\nu}(y)$ up to the backtracking loop that connects y to x, or

$$\begin{aligned} V(s,t) &= \mathrm{P}\exp\left[a\int_0^s A_\mu(x+a\tilde{s}\hat{\mu})\mathrm{d}\tilde{s}\right]\mathrm{P}\exp\left[a\int_0^t A_\mu(x+as\hat{\mu}+a\tilde{t}\hat{\nu})\mathrm{d}\tilde{t}\right], \\ \mathcal{F}_{\mu\nu}(x+as\hat{\mu}+at\hat{\nu}) &= V(s,t)F_{\mu\nu}(x+as\hat{\mu}+at\hat{\nu})V^\dagger(s,t)\,. \end{aligned} \tag{13}$$

To obtain the result of eq. (11) one now expands $\mathcal{F}_{\mu\nu}(x+as\hat{\mu}+at\hat{\nu})$ around the point x, making use of the identity

$$\partial_\mu^n\partial_\nu^m\mathcal{F}_{\mu\nu}(x) = \mathcal{D}_\mu^n\mathcal{D}_\nu^m F_{\mu\nu}(x)\,. \tag{14}$$

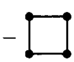
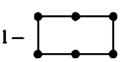
Note that the ordering of the covariant derivatives in the r.h.s. of eq. (14) is essential. Also crucial is that the path ordering $U(s,t) \equiv \mathrm{P}\exp(\int_s^t A(u)\mathrm{d}u)$ (where $A(t) = \hat{e}_\mu A_\mu(x+t\hat{e})$ for some unit vector $\hat{e}$) is compatible with the covariant derivative, i.e. $\hat{e}_\mu D_\mu(x+s\hat{e})U(s,t) = 0 = \hat{e}_\mu D_\mu(x+t\hat{e})U^\dagger(s,t)$ (in this respect we have corrected the formula in ref. [29]). Inserting the Taylor expansion of $\mathcal{F}_{\mu\nu}(x+as\hat{\mu}+at\hat{\nu})$ with respect to (s,t) in eq. (12), gives the result of eq. (11). A very useful check is that the symmetry implied by $U_{\mu\nu}(x) = U_{\nu\mu}^\dagger(x)$, not explicit at intermediate steps of the calculation, is respected by the final result.

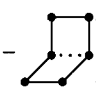
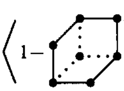
Using eq. (9,11), one finds to $\mathrm{O}(a^{10})$ for the modified action $S(\varepsilon)$

$$\begin{aligned} S(\varepsilon) = \sum_{x,\mu,\nu}\mathrm{Tr}\Bigg[&-\frac{a^4}{2}F_{\mu\nu}^2+\frac{\varepsilon a^6}{24}\Big((\mathcal{D}_\mu F_{\mu\nu}(x))^2+(\mathcal{D}_\nu F_{\mu\nu}(x))^2\Big) \\ &-\frac{\cancel{(15\varepsilon-12)a^8}}{72}\Big\{F_{\mu\nu}^4(x)+\frac{1}{30}\Big((\mathcal{D}_\mu^2F_{\mu\nu}(x))^2+(\mathcal{D}_\nu^2F_{\mu\nu}(x))^2\Big) \\ &+\frac{1}{3}\mathcal{D}_\mu^2F_{\mu\nu}(x)\mathcal{D}_\nu^2F_{\mu\nu}(x)-\frac{1}{4}(\mathcal{D}_\mu\mathcal{D}_\nu F_{\mu\nu}(x))^2\Big\}\Bigg]\,. \end{aligned} \tag{15}$$

Obviously, $S(\varepsilon = 1)$ corresponds to the Wilson action, and the sign of the leading lattice artefacts are simply reversed by changing the sign of ε. Most of the numerical results were obtained for $\varepsilon = -1$, but ε is useful in the initial cooling from a random configuration. By keeping $\varepsilon > 0$ as long as $S > 8\pi^2$, and only switching to $\varepsilon = -1$ when $S \sim 8\pi^2$, we can avoid the solution to get stuck at higher topological charges. Once we set $\varepsilon = -1$, we have yet to see an instanton fall through the lattice. We will come back to these issues when discussing the numerical results. Also note that, as $\mathrm{Tr}_{\mathrm{ad}}(U) = |\mathrm{Tr}(U)|^2 - 1$, one finds that the Wilson action in the adjoint representation (S_{ad}) satisfies $S_{\mathrm{ad}} = 4S(\varepsilon = 1) + \mathrm{O}(a^8)$ and does not allow us to change the sign of the a^6 term. The same holds for the Manton action [30] which by definition agrees to $\mathrm{O}(a^8)$ with the Wilson action.

In the past, more complicated improved actions were considered [9,31], for which we will present the result similar to eq. (15), as it allows us to predict whether or not they give rise to stable solutions [8]. It also allows comparison with earlier results by Lüscher and Weisz [31] obtained from a perturbative analysis. In the following, the coefficients in front of the c_i are to match with the definitions of ref. [31]. The averages $\langle\cdots\rangle$ are similar to the clover average above, but include now also averaging over all orientations of the loops. After some algebra one finds

$S(\{c_i\}) \equiv \sum_x \mathrm{Tr}\Big\{c_0\Big\langle 1-$  $\Big\rangle + 2c_1\Big\langle 1-$  $\Big\rangle$

$+4c_2\Big\langle 1-$  $\Big\rangle + \frac{4}{3}c_3\Big\langle 1-$  $\Big\rangle\Big\}$

$$
\begin{aligned}
&= -\frac{a^4}{2}(c_0 + 8c_1 + 16c_2 + 8c_3)\sum_{x,\mu,\nu}\mathrm{Tr}(F_{\mu\nu}^2(x)) \\
&\quad + a^6(c_2 + \frac{c_3}{3})\sum_{x,\mu,\nu,\lambda}\mathrm{Tr}(\mathcal{D}_\mu F_{\mu\lambda}(x)\mathcal{D}_\nu F_{\nu\lambda}(x)) \\
&\quad + \frac{a^6}{12}(c_0 + 20c_1 + 4c_2 - 4c_3)\sum_{x,\mu,\nu}\mathrm{Tr}(\mathcal{D}_\mu F_{\mu\nu}(x))^2 \\
&\quad + a^6\frac{c_3}{3}\sum_{x,\mu,\nu,\lambda}\mathrm{Tr}((\mathcal{D}_\mu F_{\nu\lambda})^2) + \mathrm{O}(a^8)\,. \qquad (16)
\end{aligned}
$$

One can therefore achieve tree-level improvement by choosing [31] $c_0 + 8c_1 + 16c_2 + 8c_3 = 1$, $c_0 + 20c_1 + 4c_2 - 4c_3 = 0$ and $c_2 = c_3 = 0$. Note that the condition $c_2 + c_3/3 = 0$ only applies off-shell, since on-shell

$$\sum_{x,\mu,\nu,\lambda}\mathrm{Tr}(\mathcal{D}_\mu F_{\mu\lambda}(x)\mathcal{D}_\nu F_{\nu\lambda}(x)) = 0\,.$$

Iwasaki and Yoshié [8] considered cooling for the Symanzik improved action, that is $c_0 = \frac{5}{3}$, $c_1 = -\frac{1}{12}$ and $c_{2,3} = 0$, for which the a^6 term vanishes. The a^8 term will have to be computed to settle stability. From eq. (15) one sees that the a^6 term has a definite sign. This is no longer the case for the a^8 term. The same holds for the Symanzik improved action:

$$S_{\text{Symanzik}} = \sum_{x,\mu,\nu} \mathrm{Tr}\left[-\frac{a^4}{2}F_{\mu\nu}^2(x) + \frac{a^8}{24}\left\{F_{\mu\nu}^4(x) + \frac{1}{3}\mathcal{D}_\mu^2 F_{\mu\nu}(x)\mathcal{D}_\nu^2 F_{\mu\nu}(x) - \frac{1}{4}(\mathcal{D}_\mu\mathcal{D}_\nu F_{\mu\nu}(x))^2 + \frac{4}{15}(\mathcal{D}_\mu^2 F_{\mu\nu})^2\right\}\right] + \mathrm{O}(a^{10}). \tag{17}$$

To decide in these cases if the lattice admits a stable solution (i.e. its action increases with decreasing ρ), one can compute the lattice action using explicitly the topological charge-1 instanton solution with scale ρ. Eqs. (15)–(17) are only valid as long as $a \ll \rho$ because for $\rho \sim a$ the expansion in powers of a no longer converges. For $\rho \ll L$ to a good approximation we can substitute the infinite-volume continuum instanton solution:

$$A_\mu(x) = -i\frac{\eta^a_{\mu\nu}x_\nu\sigma_a}{(x^2+\rho^2)}, \tag{18}$$

with $\eta^a_{\mu\nu}$ the self-dual 't Hooft tensor [33]. When $\rho \sim L$ the solution will of course be modified by the boundary effect. Substituting eq. (18) we find

$$S(\varepsilon) = 8\pi^2\left\{1 - \frac{\varepsilon}{5}(a/\rho)^2 - \frac{15\varepsilon - 12}{210}(a/\rho)^4 + \mathrm{O}(a/\rho)^6\right\},$$
$$S_{\text{Symanzik}} = 8\pi^2\left\{1 - \frac{17}{210}(a/\rho)^4 + \mathrm{O}(a/\rho)^6\right\}, \tag{19}$$

we thus confirm the observation of Iwasaki and Yoshié [8] that the Symanzik tree-level improved action has no stable instanton solutions. Since $S(\varepsilon = 0) = 8\pi^2\{1 + \frac{2}{35}(a/\rho)^4 + \mathrm{O}(a/\rho)^6\}$ we predict even at $\varepsilon = 0$ the lattice to have stable solutions, which we have verified for the case with twisted boundary conditions in the time direction (see below).

Iwasaki and Yoshié [8] also considered cooling for Wilson's choice [32] (W) of $c_0 = 4.376, c_1 = -0.252, c_2 = 0$ and $c_3 = -0.17$ and for (R) $c_0 = 9, c_1 = -1$ and $c_2 = c_3 = 0$. To $\mathrm{O}(a^8)$ these actions effectively correspond respectively to $\varepsilon = -2.704$ and $\varepsilon = -11$, which for the case (W) we computed by substituting the continuum instanton solution. Indeed, they see stability up to 250 sweeps in both cases.

5. Non-leading lattice artefact corrections

In presenting eq. (19) we have replaced the sum over the lattice points by an integral and ignored the fact that on the lattice the equations of motion are modified. Both effects turn out to be small, the first exponential in ρ/a, the other gives a correction to the expression for $S(\varepsilon)$ in eq. (19) proportional to

$\varepsilon^2(a/\rho)^4$ (whereas the correction to S_{Symanzik} is proportional to $(a/\rho)^8$, which also holds for $S(\varepsilon = 0)$).

We wish to compute $\sum_x f(x) = \sum_n f(na)$, for which we can use its Fourier decomposition

$$a^4 \sum_x f(x) = a^4 \sum_{n\in\mathbb{Z}^4} \sum_k e^{ik\cdot na} \tilde{f}(k)$$
$$= a^4 N_s^3 N_t \sum_{p\in\mathbb{Z}^4} \tilde{f}(\frac{2\pi p}{a}) = \sum_{p\in\mathbb{Z}^4} \int e^{-2\pi i p\cdot x/a} f(x) d^4x \,. \tag{20}$$

The terms with $p \neq 0$ give the error one makes, when replacing the lattice sum by an integral. For $a \ll \rho \ll L$ and $f(x) = -\frac{1}{2}\text{Tr}(F_{\mu\nu}^2(x + x_0))$ one finds explicitly (using eq. (18))

$$-\frac{a^4}{2} \sum_{x,\mu,\nu} \text{Tr}(F_{\mu\nu}^2(x + x_0))$$
$$= 8\pi^2 \left[1 + \sum_{p\in\mathbb{Z}^4\setminus\{0\}} 2\pi^2 p^2 (\rho/a)^2 \cos(2\pi p \cdot x_0/a) K_2(2\pi|p|\rho/a)\right]$$
$$= 8\pi^2 \left[1 - 8\pi^2(\rho/a)^{3/2} e^{-2\pi\rho/a}(1 + O(a/\rho))\right] , \tag{21}$$

(with K_2 the modified Bessel function [34]). Here we have taken x_0 to coincide with a point on the *dual* lattice, $2x_0^\mu = a$ for all μ, as this minimizes the action.

To estimate the shift in the equations of motion due to the lattice artefacts, we again consider $\rho \gg a$, such that the action can, in a good approximation, be given by

$$\tilde{S}(\varepsilon, A_\mu) = \sum_{\mu,\nu} \int d^4x \left\{ -\frac{1}{2}\text{Tr}(F_{\mu\nu}^2(x)) + \frac{\varepsilon a^2}{12}\text{Tr}((\mathcal{D}_\mu F_{\mu\nu}(x))^2) \right.$$
$$-a^4 \frac{(15\varepsilon - 12)}{72} \text{Tr} \left[F_{\mu\nu}^4(x) + \frac{1}{15}(\mathcal{D}_\mu^2 F_{\mu\nu}(x))^2 \right.$$
$$\left.\left. + \frac{1}{3}\mathcal{D}_\mu^2 F_{\mu\nu}(x) \mathcal{D}_\nu^2 F_{\mu\nu}(x) - \frac{1}{4}(\mathcal{D}_\mu \mathcal{D}_\nu F_{\mu\nu}(x))^2 \right] \right\} + O(a^6), \tag{22}$$

which implies the equations of motion

$$\sum_\nu \mathcal{D}_\nu F_{\nu\mu} = \varepsilon a^2 H_\mu + O(a^4),$$
$$H_\mu \equiv -\sum_\nu (\tfrac{1}{12}\mathcal{D}_\nu^3 F_{\nu\mu} + \tfrac{1}{6}[F_{\mu\nu}, \mathcal{D}_\mu F_{\mu\nu}] + \tfrac{1}{12}\mathcal{D}_\mu^2 \mathcal{D}_\nu F_{\nu\mu}). \tag{23}$$

As eq. (22) breaks the scale invariance, there will in general not be solutions close to eq. (18). Variation with respect to ρ no longer leaves the action invariant. Still, since this variation corresponds to a near zero-mode, it makes sense to expect quasi-stability under cooling. The action changes only slowly in the direction of this near zero-mode but is predominantly lowered in those directions that leave the curvature square integrable and are spanned by the non-zero modes of the quadratic fluctuation operator for the action, which in the background gauge corresponds to

$$\mathcal{M}_{\lambda\sigma} = \delta_{\lambda\sigma}\mathcal{D}_\mu^2 + 2\,\mathrm{ad}\,F_{\lambda\sigma}\,. \tag{24}$$

If $\mathcal{P}$ is the projection operator on the normalizable non-zero modes of $\mathcal{M}$ one has (at $\rho = 1$)

$$\begin{aligned} A_\mu^{(\varepsilon)} &= A_\mu^{(0)} + \varepsilon a^2 \mathcal{P}\mathcal{M}_{\mu\nu}^{-1}\mathcal{P}H_\nu + \mathrm{O}(a^4) \equiv A_\mu^{(0)} + \varepsilon a^2 A_\mu^{(1)} + \mathrm{O}(a^4)\,, \\ H_\mu(x) &= \frac{16}{(1+x^2)^5}\psi_\mu^{(3/2,5/2)}(x) + \frac{8}{(1+x^2)^5}\psi_\mu^{(1/2,1/2)}(x)\,, \end{aligned} \tag{25}$$

where $H_\mu(x)$ is evaluated by substituting for $A_\mu^{(0)}(x)$ the continuum solution $A_\mu(x)$ given in eq. (18). For convenience we introduced the quantities $\psi_\mu^{(l,j)}(x)$:

$$\begin{aligned} \psi_\mu^{(1/2,1/2)}(x) &= i\sum_{\nu,a}\eta_{\mu\nu}^a x_\nu \sigma_a\,, \quad \psi_\mu^{(3/2,3/2)}(x) = i\sum_{\nu,a}\eta_{\mu\nu}^a x_\nu \sigma_a (x^2 - 6x_\mu^2)\,, \\ \psi_\mu^{(3/2,5/2)}(x) &= i\sum_{\nu,a}\eta_{\mu\nu}^a x_\nu \sigma_a (3x^2 - 3x_\mu^2 - 5x_\nu^2)\,, \end{aligned} \tag{26}$$

which are eigenfunctions of the angular momentum operators $\boldsymbol{L}_1^2$ and $\boldsymbol{J}^2$ (as defined in ref. [33] $L_1^a = -\frac{1}{2}i\eta_{\mu\nu}^a x_\mu \partial_\nu$, $J^a = L_1^a + \mathrm{ad}(\sigma_a/2)$). To compute $\mathcal{M}_{\mu\nu}^{-1}\mathcal{P}H_\nu$ one can use for $\mathcal{M}_{\mu\nu}^{-1}$ the explicit expression [35]

$$\mathcal{M}_{\mu\nu}^{-1} \equiv \bar{\eta}_{\mu\lambda}^a \mathcal{D}_\lambda (\mathcal{D}_\alpha^{-2}) \bar{\eta}_{\nu\sigma}^a \mathcal{D}_\sigma$$

(in the gauge $\mathcal{D}_\mu(\mathcal{P}H_\mu) = 0$). The result, which can be verified by applying $\mathcal{M}_{\mu\nu}$, is found to be

$$\begin{aligned} \mathcal{M}_{\mu\nu}^{-1}\mathcal{P}H_\nu = &\left(\frac{\log(1+x^2)}{5(1+x^2)^2} - \frac{1}{3(1+x^2)^3} + \frac{1}{10(1+x^2)}\right)\psi_\mu^{(1/2,1/2)}(x) \\ &+ \left(\frac{2\log(1+x^2)}{5x^8(1+x^2)^2} - \frac{6+3x^2-x^4}{15x^6(1+x^2)^3}\right)\psi_\mu^{(3/2,3/2)}(x) \\ &+ \left(\frac{2(3+5x^2)\log(1+x^2)}{5x^8(1+x^2)^2} - \frac{6+13x^2+4x^4}{5x^6(1+x^2)^3}\right)\psi_\mu^{(3/2,5/2)}(x)\,. \end{aligned} \tag{27}$$

The algebraic manipulation program Mathematica [36] was useful in obtaining and checking these results. Despite its appearance, this result is regular at $x \to 0$. However, it contains a non-normalizable deformation (since $\psi_\mu^{(1/2,1/2)}(x)/(1+x^2) = -A_\mu^{(0)}(x)$), which would make the action diverge and should be removed by projecting on normalizable deformations:

$$A_\mu^{(1)} = \mathcal{P}\mathcal{M}_{\mu\nu}^{-1}\mathcal{P}H_\nu = \mathcal{M}_{\mu\nu}^{-1}\mathcal{P}H_\nu - \frac{23+17x^2}{60(1+x^2)^2}\psi_\mu^{(1/2,1/2)}(x),$$

$$\mathcal{M}_{\mu\nu}A_\nu^{(1)} = H_\mu - \frac{8\psi_\mu^{(1/2,1/2)}(x)}{5(1+x^2)^3}. \tag{28}$$

One easily verifies that $A_\mu^{(1)}$ and $\mathcal{M}_{\mu\nu}A_\nu^{(1)}$ are both square integrable and orthogonal to the zero-mode $\partial A_\mu^{(0)}/\partial\rho|_{\rho=1} = 2\psi_\mu^{(1/2,1/2)}(x)/(1+x^2)^2$.

We can now substitute $A_\mu^{(\varepsilon)} = A_\mu^{(0)} + \varepsilon a^2 A_\mu^{(1)} + \mathrm{O}(a^4)$ in eq. (15), to obtain the shift in the action. It will be useful for verifying eq. (32) if we evaluate

$$\begin{aligned}\tilde{S}(\tilde{\varepsilon}, A_\mu^{(\varepsilon)}) &= \tilde{S}(\tilde{\varepsilon}, A_\mu^{(0)}) - 2\varepsilon\tilde{\varepsilon}a^4\int d^4x\,\mathrm{Tr}(A_\mu^{(1)}(x)H_\mu(x)) \\ &\quad + \varepsilon^2 a^4 \int d^4x\,\mathrm{Tr}(A_\mu^{(1)}(x)\mathcal{M}_{\mu\nu}A_\nu^{(1)}(x)) + \mathrm{O}(a^6).\end{aligned} \tag{29}$$

The equations of motion for $A_\mu^{(0)}$ where used to simplify the term linear in the shift $A_\mu^{(\varepsilon)} - A_\mu^{(0)} = \varepsilon a^2 A_\mu^{(1)} + \mathrm{O}(a^4)$, which makes it evident that the $\mathrm{O}(a^4)$ term in $A_\mu^{(\varepsilon)}$ will only contribute $\mathrm{O}(a^6)$ to eq. (29). To evaluate the action used for cooling, one simply equates $\tilde{\varepsilon}$ to ε. Evaluating the integrals, reintroducing the ρ dependence using trivial dimensional arguments, gives

$$\begin{aligned}\tilde{S}(\tilde{\varepsilon}, A_\mu^{(\varepsilon)}) &= 8\pi^2\left\{1 - \frac{\tilde{\varepsilon}}{5}\left(\frac{a}{\rho}\right)^2 - \left[\frac{15\tilde{\varepsilon}-12}{210} + \frac{284\varepsilon\tilde{\varepsilon}}{2625} - \frac{179\varepsilon^2}{5250}\right]\left(\frac{a}{\rho}\right)^4\right. \\ &\quad \left. + \mathrm{O}\left(\frac{a}{\rho}\right)^6\right\}.\end{aligned} \tag{30}$$

At $\varepsilon = \tilde{\varepsilon} = -1$ the shift due to the modified equations of motion in the $(a/\rho)^4$ term is 58%.

Strictly speaking our expression for the ρ-dependence of the lattice action is only valid for $\rho/a \gtrsim 2$ and $\rho \ll L$, since we are using the continuum infinite volume solution as the zero-order approximation. But even for $\rho \sim L/2$ it is not unreasonable to expect the order of magnitude of the corrections to be given by eqs. (21) and (30).

6. Numerical results and discussion

This section discusses the numerical results obtained as described in section 3, mainly to illustrate the viability of our ideas. A more detailed and careful analysis will be left for a future publication. So far we have worked mainly on lattices of size $N_s^3 \times N_t$, with $N_s = 7$ or 8 and with $N_t = 3N_s$ to $4N_s$. At $\varepsilon = -1$ (see eq. (8)) we settle to an action near $8\pi^2$, and we have seen stability for up to 6000 sweeps. The same is true for $\varepsilon = 0$ in the presence of a twist (but not without twist,

TABLE 1

Numerical results obtained by cooling with $S(\varepsilon)$ and twist $n_{0i} = (1,1,1)$.

$N_s \times N_t$	ε	$S_{1\times1}$	$S_{1\times2}$	$S_{2\times2}$	$S_{1\times3}$
$7^3 \times 21$	-1	0.982591	0.957050	0.928823	0.918105
$7^3 \times 21$	0	0.982287	0.956437	0.927908	0.917109
$8^3 \times 24$	-1	0.986720	0.967122	0.945887	0.936619
$8^3 \times 24$	0	0.986529	0.966736	0.945310	0.935976

where our configuration ultimately decays to the vacuum at $\varepsilon = 0$, but note that in that case there are no regular instanton solutions). Apart from the total action we compute separately the sum over the $n \times m$ plaquettes, denoted by $S_{n\times m}$ and averaging over the two orientations if $n \neq m$. $S_{n\times m}$ is normalized by dividing by $8\pi^2 n^2 m^2$, such that for an infinite lattice and $\rho/a \to \infty$, $S_{n\times m} \to 1$. When we perform cooling with the action of eq. (8) we should take $A_\mu = A_\mu^{(0)} + \varepsilon a^2 A_\mu^{(1)}$ in calculating $S_{n\times m}$. Eq. (11) for $S_{1\times1}$ easily leads to the general result for $S_{n\times m}$ by inserting for each index $\mu(\nu)$ a factor $n(m)$. Together with eq. (29) one deduces, that to $O(a^6)$

$$\begin{aligned} S_{n\times m} &= 1 - \frac{(n^2+m^2)}{2}\alpha a^2 \\ &\quad - \left(m^2 n^2 \beta_1 - \frac{m^4+n^4}{2}\beta_2 + \frac{n^2+m^2}{2}\varepsilon\gamma - \varepsilon^2\delta\right)a^4, \end{aligned} \tag{31}$$

up to the discretization error implied by eq. (21), which for the lattices we are considering can be estimated to be not bigger than 10^{-6}. This formula holds for sufficiently smooth configurations, i.e. $\alpha a^2 \ll 1$, even if the configuration has non-vanishing action over the entire spatial volume. It is these configurations that are of interest to us and which deviate considerably from localized instantons (eq. (18)) for which $\rho \ll L$. From eqs. (19, 30) we easily deduce for those localized instantons the results

$$\begin{aligned} &\alpha a^2 = \frac{1}{5}\left(\frac{a}{\rho}\right)^2, \qquad \beta_1 a^4 = \frac{29}{630}\left(\frac{a}{\rho}\right)^4, \qquad \beta_2 a^4 = \frac{2}{63}\left(\frac{a}{\rho}\right)^4, \\ &\gamma a^4 = \frac{284}{2625}\left(\frac{a}{\rho}\right)^4, \qquad \delta a^4 = \frac{179}{5250}\left(\frac{a}{\rho}\right)^4. \end{aligned} \tag{32}$$

From the numerical results we have obtained $S_{1\times1}$, $S_{1\times2}$, $S_{2\times2}$ and $S_{1\times3}$ on two lattices of size respectively $7^3 \times 21$ and $8^3 \times 24$, for $\varepsilon = 0$ and $\varepsilon = -1$ with a twist $n_{0i} = (1,1,1)$ (see table 1). From these we extract the coefficients in eq. (31), whose values are summarized in table 2 (the error due to neglecting the $O(a^6)$ term is of the order of $(n^6 + m^6)(\alpha a^2)^3$).

It is interesting to analyse the untwisted case in more detail to illustrate the difficulty in having self-dual solutions at finite T. In fig. 1a we plot the total elec-

TABLE 2

Coefficients appearing in eq. (31) extracted from the numerical results in table 1, using $S_{1\times 1}$, $S_{2\times 2}$ and $S_{1\times 2}$ (the latter at $\varepsilon = -1$ only).

$N_s \times N_t$	αa^2	β_1/α^2	β_2/α^2	γ/α^2	δ/α^2
$7^3 \times 21$	0.01761	0.96	0.63	0.66	0.32
$8^3 \times 24$	0.01340	1.01	0.64	0.71	0.35
$\rho \ll L$	$0.2(a/\rho)^2$	1.15	0.79	2.70	0.85

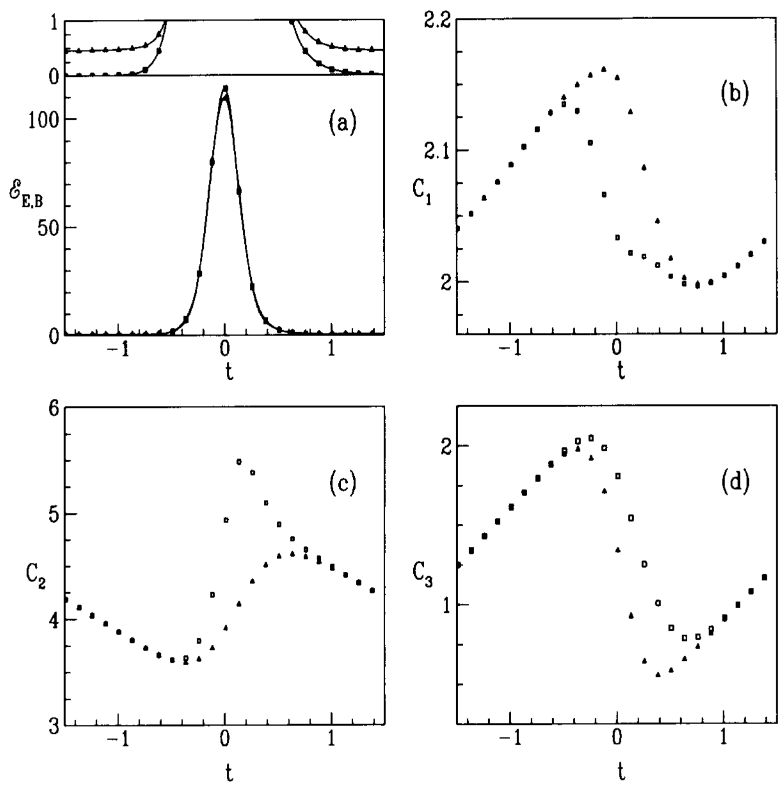


Fig. 1. Numerical results (after scaling appropriately with N_s) for the case of an $8^3 \times 24$ lattice without twist, obtained from over-improved cooling at $\varepsilon = -1$. In (a) the electric ($\mathcal{E}_E(t)$ triangles) and magnetic ($\mathcal{E}_B(t)$ squares) energies are plotted. In the upper part of this figure the tails are plotted at an enlarged scale. In (b-d) are plotted $C_i(t) \equiv 2a\cos(P_i(t))$ through two distinct spatial points on the lattice.

tric and magnetic energies $\mathcal{E}_{E,B}(t)$, in fig. 1b the Polyakov line $P_1(t)$ through two particular points x and similarly for $P_{2,3}(t)$ in figs. 1c,d. We see two features that are intimately related. First, where $\mathcal{E}_B(t) \to 0$, the electric energy $\mathcal{E}_E(t) \to$ const. Second for the same t values where this occurs $C_i(t)$ $(P_i(t) = \cos(C_i(t)/2))$ is

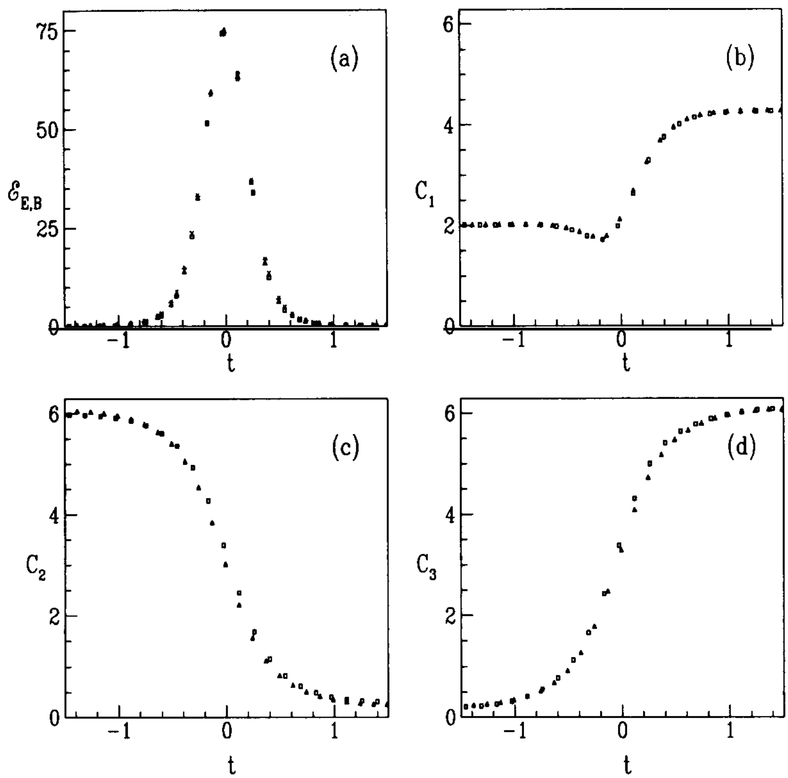


Fig. 2. Numerical results (after scaling appropriately with N_s) for the cases of a $7^3 \times 21$ (squares) and an $8^3 \times 24$ (triangles) lattice with twist $n_{0i} = (1, 1, 1)$, obtained from over-improved cooling at $\varepsilon = -1$. Fig. 2a contains four data sets. Two for $\mathcal{E}_E(t)$ with the above mentioned symbols and two (crosses for $N_s = 7$ and stars for $N_s = 8$) for $\mathcal{E}_B(t)$. Figs. 2b–d exhibit $C_i(t)$, through the spatial lattice point with maximal E_1^2 at $t = 0$.

linear in t and x independent. These are precisely the equations of motion when restricting to the vacuum valley. Classically motion on this valley, which itself has the geometry of a three-torus, is free. On the lattice this motion is described by the action

$$\sum_t 4N_s^3 \left[1 - \cos\left(\frac{C_k(t+1) - C_k(t)}{2N_s}\right)\right] . \tag{33}$$

One easily checks that the values of $C_k(t+1) - C_k(t)$ obtained from figs. 1b–d quite accurately reproduce through eq. (33) the value for $\mathcal{E}_E(t)$. Clearly the electric tail destroys the self-duality. Suppose that at $T \to \infty$ the solution describes tunnelling from C_i^- to C_i^+ and $C_i^+ \neq C_i^-$, then at finite T the periodic boundary conditions force $C_i(t)$ to linearly interpolate between C_i^+ and C_i^- over a time $T - T_0$, if T_0 is the time interval over which $\mathcal{E}_{\rm B}(t) \neq 0$. Thus the action, even in the continuum, would be bigger than $8\pi^2$ by a number proportional to $1/(T-T_0)$ except when there are solutions with $C_i^- = C_i^+$ for $T \to \infty$. These can certainly not be excluded, in particular as $C_i^+ = C_i^- = \pi$ is compatible with a twist $n_{0i} = 1$, but if these are very localized instantons, the lattice artefacts might make their action so big, that the lattice will prefer the least localized solutions with $C_i^+ \neq C_i^-$. If T is not big enough the lattice will find a compromise between these two cases. There are indications that the largest instanton prefers $C_i^+ \neq C_i^-$ and from the numerical results with twist $n_{0i} = (1,1,1)$ presented in fig. 2, the preferred values seem to be such that two of the C_i go from 0 to 2π and one goes from $2\pi/3$ to $4\pi/3$. We compare (after appropriate scaling with $N_{\rm s}$) for $N_{\rm s} = 7$ and 8, using over-improved cooling at $\varepsilon = -1$, in fig.2a the electric and magnetic energy profiles, and in figs. 2b–d the values of $C_i \equiv 2a\cos(P_i)$, at the spatial lattice point with maximal energy (to be precise, with maximal E_1^2). From this we deduce $\mathcal{E}_{\rm E} = \mathcal{E}_{\rm B}$ to a high accuracy, consistent with self-duality, and the excellent scaling with $N_{\rm s}$. The results in fig. 2 are obtained after roughly 6600 cooling sweeps, which is necessary since the dependence of the lattice action on $C_i^{\pm}$ is rather weak (at $\varepsilon = 0$ too weak to observe) and the configuration only slowly reaches the minimum of the lattice action. We have verified that the approach to this minimum is exponential, as is illustrated in fig. 3, where we plot the total action and the maximum of $\mathcal{E}_{\rm B}(t)$ (i.e. $\mathcal{E}_{\rm B}(0)$) as a function of the number of cooling sweeps. We see indeed that the maximal energy along the tunnelling path decreases under cooling, which is mainly due to the increasing size, as otherwise the action should depend more strongly on the number of cooling sweeps. (For the Wilson action one sees a dramatic increase of $\mathcal{E}_{\rm B}(0)$ under cooling, until the action suddenly drops to zero.) With boundary conditions that fix the link variables at $t = 0$ and $t = T$ to the vacuum configurations, the approach to the minimum action is much faster.

Elsewhere we will publish a more detailed analysis of the scaling properties, as well as testing our conjecture to be able to find a sphaleron. Also numerical results with fixed boundary conditions, that allow us to investigate if solutions exist for arbitrary $C_i^{\pm}$ will be presented elsewhere. This paper mainly served the purpose to describe the formalism, and demonstrate the large amount of control obtained in this way in studying instanton solutions on a torus.

It would be interesting to repeat this analysis for the two-dimensional O(3) model, for which the instantons on a torus are exactly known [37], in the light of the "perfect" lattice action recently considered by Hasenfratz and Niedermayer [38]. But as we have shown, appropriate deviations from a "perfect" action can be quite helpful.

Finally, over-improvement might be an efficient tool to measure the topological susceptibility, as the action to generate a statistical ensemble need not be the same as the one used to measure the topological charge.

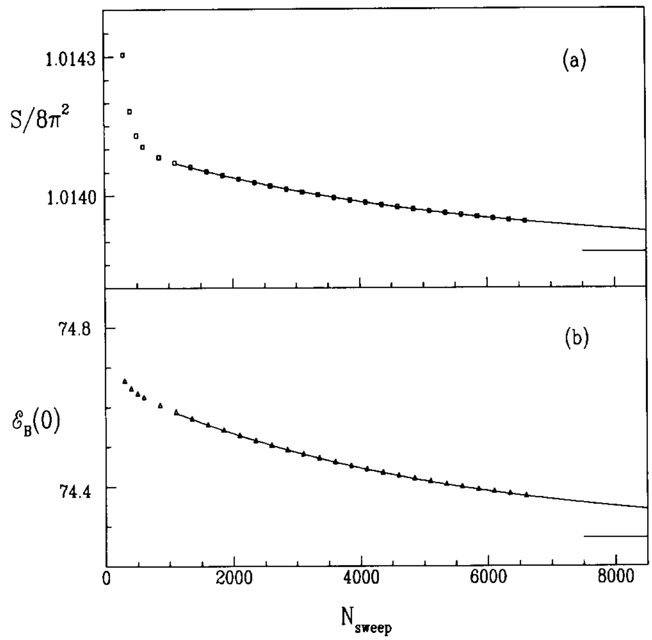


Fig. 3. The history of the action $S(\varepsilon = -1)$ and the maximal magnetic energy $\mathcal{E}_B(t = 0)$ as a function of the number of cooling sweeps for an $8^3 \times 24$ lattice with twist $n_{0i} = (1, 1, 1)$, together with their exponential fits. The short lines on the right indicate the asymptotic values following from these fits.

We have benefited from discussions with Wolfgang Bock, Maarten Golterman, Peter Hasenfratz, Jim Hetrick, Arjan Hulsebos, Ferenc Niedermayer, Erhard Seiler and Peter Weisz. In particular we thank Peter Braam and Jan Smit for discussions on respectively the moduli spaces and improved actions. One of us (P.v.B.) wishes to take this opportunity to thank Erhard Seiler and Peter Weisz for hospitality at the Max Planck Institute in Munich, Dan Freedman and Ken Johnson for hospitality at MIT and Tony González-Arroyo for hospitality at UAM in Madrid.

This work was supported in part by grant number AEN 90-0272, financed by CICYT, and by grants from Stichting voor Fundamenteel Onderzoek der Materie (FOM) and Stichting Nationale Computer Faciliteiten (NCF) for use of the CRAY Y-MP at SARA. M.G.P. was supported by a Human Capital and Mobility EC fellowship.

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