



Recent results on self-dual configurations on the torus*

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We review the recent progress on our understanding of self-dual $SU(N)$ Yang-Mills configurations on the torus.

1. Motivation

The purpose of this paper is to briefly review the progress made during the last year in the investigation of $SU(N)$ self-dual Yang-Mills (YM) configurations on the 4-dimensional torus (T^4). It is well-known that there is a procedure (the ADHM construction [1]) to construct all self-dual configurations on the sphere. Conversely, for the torus case not even the simplest fully non-abelian solution is known analytically (abelian-like solutions are known for some torus sizes). Constructing them provides a challenge for both physicists and mathematicians.

From the Physics point of view, compactification on the sphere amounts to the condition of finite action implying that the configuration approaches a pure gauge at infinity. This is not a physically necessary condition even at a purely classical level. Since the action is an extensive quantity it makes more sense to demand finiteness of the action density. In the Confinement regime, it is reasonable to expect that typical configurations are different from the classical vacuum almost everywhere. On the other hand, configurations on the torus can be looked at as periodic configurations on R^4 having finite action density. Although periodicity is also an unphysical feature for the dominant Yang-Mills configurations to have, knowledge of these configurations might help us to understand better some structures which might be present in the vacuum.

Besides, there are some cases in which one deals with periodicity in some of the variables. This

can be seen as formulations on the non-compact manifolds $T^n \times R^{4-n}$. The case $n = 1$ occurs naturally when studying field theory at finite temperature. The case $n = 3$ has been studied in relation with the Hamiltonian formulation for YM fields on the torus. Recently [2], it has been seen how the $n = 2$ case is relevant in constructing $SU(N)$ YM vortex-like configurations in R^4 . It is an interesting question to investigate how the T^4 configurations are related to these non-compact manifold ones.

2. Introduction

Yang-Mills fields on the torus are classified by the topological charge Q and by the twist sectors (see Ref. [3] for an introduction to the subject and a review of older results). The latter are a discrete number of sectors labelled by two 3-vectors of integers modulo N (\vec{k} and \vec{m}). The possible values of the topological charge are restricted by twist:

$$Q = -\frac{\vec{k} \cdot \vec{m}}{N} + \text{integer} \quad . \quad (1)$$

Orthogonal twists are those for which $\vec{k} \cdot \vec{m} = 0 \bmod N$ (the rest are called non-orthogonal). Only in this case the topological charge is an integer (and the action a multiple of $8\pi^2$).

Self-dual solutions, if existing, form a manifold whose dimensionality (up to gauge transformations) is given by $4QN$. Four of these modes correspond to space-time translations of the solution. Existence and non-existence has been proved in some cases. Particularly relevant is the non-existence of $Q = 1$ self-dual configurations on the torus without twist [4] ($\vec{k} = \vec{m} = 0 \bmod N$).

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Apart from the afore-mentioned quasi-abelian solutions, most of what is known about these solutions comes from numerical studies on the lattice. There is, however, an important duality (involutive) transformation –the Nahm transform [5]– which maps $SU(N)$ self-dual configurations on the torus with topological charge Q , onto $SU(Q)$ self-dual configurations on the dual torus with topological charge N . Unfortunately, besides its role in the proof of Ref. [4], little use has been made of this property to increase our knowledge on these configurations. Part of the progress we report, has to do with fixing this state of affairs.

The study of self-dual configurations depends on the group rank, twist, topological charge and torus size. For simplicity the study has focused on $SU(2)$ and low values of the topological charge ($Q = \frac{1}{2}, 1$). This forces non-zero twist. For obvious symmetry reasons the study has centered on hypercubical tori of size $l_s^3 \times l_t$. For large and small aspect ratios l_s/l_t one can study and exploit the connection with $R^3 \times S_1$ and $T^3 \times R$ configurations. Now we will summarise what was known of these configurations before.

$l_t/l_s \gg 1$ [6]: For $Q = \frac{1}{2}$ (non-orthogonal twist) the solution is unique up to translations. The configuration is exponentially localised in time and as l_t/l_s goes to infinity approaches a configuration known as the twisted instanton. For $Q = 1$ the description of the resulting configuration depends on twist. For $\vec{m} \neq 0$ and $\vec{k} = 0 \bmod N = 2$ (Space-like twist) the configuration is generically described by 2 twisted instantons separated in time. The 2 space-time locations describe the 8 parameters of the moduli space. For $Q = 1$, $\vec{k} \neq 0$ and $\vec{m} = 0 \bmod 2$ (Time-like twist) one gets a family of (exponentially localised in time) configurations which as l_t/l_s goes to infinity can be parametrised by the holonomies: the spatial Polyakov loops at infinite time.

$l_s/l_t \gg 1$: In this case the $R^3 \times S_1$ configurations with $Q = 1$ are known analytically [7]. Generically, they are given by 2 lumps which for large separations are simply BPS monopoles of various masses, fixed by the holonomy of the configuration (the time-like Polyakov loop at spatial infinity). We recently studied, by lattice methods, the large l_s/l_t configurations on the torus [8].

These configurations neatly approach the analytic calorons with some restrictions on the moduli space. For $Q = \frac{1}{2}$ a single fixed mass periodic BPS monopole is obtained. For $Q = 1$ and time-like twist one obtains in each cell a couple of equal mass constituent monopoles with arbitrary locations. Conversely for $Q = 1$ and space-like twist one gets a couple of variable mass monopoles with fixed relative positions.

3. New results

In the last year there have been several developments which we will now list:

- A numerical method, based on lattice gauge theories, has been developed to implement the NT numerically [9]. It allows to obtain the NT of lattice configurations approximating the self-dual continuum ones. The method has proved quite precise and stable.
- It has been shown how to extend the definition of the Nahm transform to the non-zero twist case [10]. The extension preserves the main properties of the original NT. The transform of an $SU(N)$ self-dual configuration on the torus with topological charge Q , is now an $SU(QN_0)$ self-dual configuration on some Nahm-dual torus with topological charge N/N_0 . The integer N_0 depends on twist and equals 1 for zero twist.
- An analysis of the NT of $Q = 1$ self-dual configurations on $T^3 \times R$ for twisted boundary conditions in time has been performed [11]. Although, the original self-dual configuration is not known, its NT is known to be an abelian field in T^3 , which is self-dual everywhere except at certain point-like singularities. These singularities act as dyonic sources and their location is determined by the holonomies (spatial Polyakov loop at infinite time) of the original $T^3 \times R$ configuration. With this information one is able to construct this Nahm-dual abelian field everywhere.

- The numerical investigation of the periodic calorons mentioned before [8].

Equipped with this new information and techniques we recently studied [12] the action of the NT on the torus configurations under concern. One finds that configurations with large and small l_s/l_t are mapped onto each other by the NT. Furthermore, for $Q=1$ the time-like or space-like nature of the twist is preserved by the transformation. The results can be summarised as follows:

- The $Q = \frac{1}{2}$ twisted instanton maps onto the $Q = \frac{1}{2}$ periodic caloron. This can be tested by the numerical NT and shows remarkable precision (see fig. 1 of the second ref. in [9]).
- For $Q = 1$ and time-like twist we see that the approximate holonomy of the $l_t/l_s \gg 1$ periodic instanton, maps onto the relative position of two equal mass constituent monopoles. Taking the limit $l_t \rightarrow \infty$ (l_s fixed) we see that the dyonic singularities of the NT of $T^3 \times R$ confs, are nothing but BPS monopoles with the non-abelian cores (of size $1/l_t$) shrunk to 0.
- For $Q = 1$ and space-like twist, the holonomy of the large l_t/l_s configuration is fixed, which explains the fixed relative position of the constituent monopoles of the Nahm transform. Furthermore, the time distance between the 2 twisted instantons of this configuration maps onto the holonomy (and hence the monopole masses) of the periodic caloron configuration. This is clearly depicted in Fig. 1.

For details the reader is referred to [12]

In summary, all the known information is nicely linked non-trivially together by the NT. A general pattern mapping approximate holonomies to lump positions emerges from our study.

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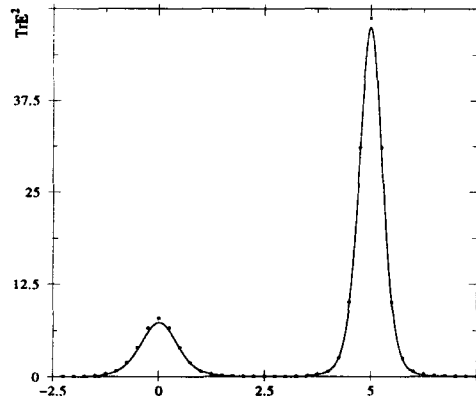


Figure 1. Comparison of the NT of an $l_t/l_s \gg 1$ configuration with the analytic caloron profile for the predicted unequal mass constituent monopoles (no free parameters).

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