LIGHTLIKE SINGULARITIES IN COMPACTIFIED SUPERGRAVITY

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We discuss the (causal) structure of a recently found black hole solution of compactified $d = 11$ supergravity. It is shown that the singularity is in fact lightlike and coincides with the horizon. Consequences are that the Hawking temperature is undetermined and that there is no other universe connected to the singularity.

In a previous paper in collaboration with P. van Nieuwenhuizen [1], we have constructed a number of black hole solutions of $d = 11$ supergravity compactified to four dimensions over a seven-sphere. These solutions tend asymptotically to the well known Freund–Rubin [2] or Englert [3] solutions but near the core of the black hole they differ substantially from the conventional Schwarzschild or Reissner–Nordström solutions. In this letter we discuss in detail the core structure of the solution which tends asymptotically to the Freund–Rubin solution (case II in ref. [1]) and reveal some novel features, notably the existence of a lightlike singularity.

As a starting point the following ansatz for the metric was considered:

$$-\mathrm{d}s^2 = -B(r)\,\mathrm{d}t^2 + A(r)\,\mathrm{d}r^2 + r^2(\mathrm{d}\Omega_2)^2 + R(r)^2(\mathrm{d}\Omega_7)^2\,,$$

$$(\mathrm{d}\Omega_n)^2 = \mathrm{d}\psi_n^2 + \sin^2\psi_n(\mathrm{d}\Omega_{n-1})^2\,. \qquad (1)$$

It describes a static metric with $\mathrm{R}_1 \times \mathrm{SO}_3 \times \mathrm{SO}_8$ symmetry. For the three-index photon field A_{MNP} of $d = 11$ supergravity we choose a Freund–Rubin ansatz [2]:

$$F_{mnpq} = \mathrm{i}b(r)\,\epsilon_{mnpq}\,, \qquad (2)$$

where b is allowed to depend on r and m, n, p and q are four-dimensional flat indices, the other components of F are put equal to zero. For large r we want the extra seven-dimensional space to be compact with radius R_∞, which implies that b and R have to go to constants related by:

$$b_\infty^2 = \tfrac{1}{32}R_\infty^{-2}\,. \qquad (3)$$

The only nontrivial Maxwell equation is integrable and yields the relation bR^7 = constant. So that after rescaling the variables, such that $R_\infty = 1$, one has that (the sign of b is irrelevant):

$$b^2 = \tfrac{1}{32}R^{-14}\,. \qquad (4)$$

This allows us to eliminate b from the Einstein equations. The Einstein equations provide us with three independent equations for A, B and R. For reasons which will become clear shortly we choose the following particular combinations:

$$A'/A + B'/B = (7rR''/R)/(1 + 7rR'/2R)\,, \qquad (5)$$

$$A'/A - B'/B = (2/r)(1-A) + 14R'/R - 24rA/R^{14}\,, \qquad (6)$$

$$A = r[r(\ln R)']' \cdot [-r(\ln R)'(1 + 12r^2R^{-14}) + 6r^2(R^{-2} - R^{-14})]^{-1}\,. \qquad (7)$$

Two remarks are in order:

(i) From (5) and (6) B'/B may be eliminated. Combining the remaining equation with (7) gives a single third order differential equation for R, which is rather easy to handle numerically.

(ii) Eq. (5) is interesting. For the ordinary Schwarzschild or Reissner–Nordström solution the right-hand side vanishes, implying that AB = constant. This relation has an important consequence for the horizon ($B = 0$) and singularity ($A = 0$) structure, namely that they cannot coincide. It is exactly because of the fact that in the present case the right-hand side of (5) does not vanish that new possibilities do arise.

An expansion around the asymptotic Freund–Rubin solution reveals that the solution contains two free parameters, apart from the overall scale. Namely a mass parameter m and a "charge" q (we will comment on the quotation marks later)

$$R = 1 + qr^{-6}[1 - \tfrac{15}{44}r^{-2} + \tfrac{1}{2}mr^{-3} + O(r^{-4})] , \quad (8)$$

$$A = [1-(2m/r)+4r^2]^{-1}\{1-49qr^{-6}[1+O(r^{-2})]\}, \quad (9)$$

$$B = [1 - (2m/r) + 4r^2] \times \{1 - \tfrac{7}{4}qr^{-6}[1 - (3m/r) + O(r^{-2})]\} . \quad (10)$$

For $m = 0$ one can even calculate the linear term in q for R exactly:

$$\partial R/\partial q\,|_{m=0,q=0} = (1 + 4r^2)^{-3}F(3,3,\tfrac{11}{2};(1+4r^2)^{-1}) , \quad (11)$$

where F is the hypergeometric function [4]. One simply uses the fact that for $q \to 0$, A is known, then (7) reduces to a linear second order differential equation for $\partial R/\partial q\,|_{q=0}$.

Note that the $4r^2$ in the prefactor of A and B comes from the cosmological constant which is induced by the nonvanishing asymptotic value of b in (3). It is in fact possible to make the physical cosmological constant zero, by adding a (fine tuned) cosmological constant $\Lambda = 4$ to the eleven-dimensional lagrangian [1]. This term breaks supersymmetry and also alters the asymptotics (8)–(10) in an essential way: instead of power corrections one obtains exponentials:

$$R = 1 + qr^{-(1+4m)}\exp(-4r)[1 + O(m^2 + 1)/r] . \quad (12)$$

Also here we can find an exact expression for $m = 0$:

$$\partial R/\partial q\,|_{m=0,q=0} = r^{-1}\exp(-4r) . \quad (13)$$

Of course the leading asymptotic form for A and B is given by $B = A^{-1} = 1 - 2m/r$.

We have numerically integrated the system (5)–(7) for various values of the parameters m and q. In fig. 1 we have plotted the solution with $m = 100$ and $q = 500$. This example is representative for the generic case where $q > 0$ [‡1]. In fact for decreasing values of q, $A(r)$ becomes more and more peaked approaching the solution $A_{q=0}$ which has a singularity (horizon) as indi-

[‡1] The case $q < 0$ exhibits a singularity in A at a finite value of r. This case is presently under investigation.

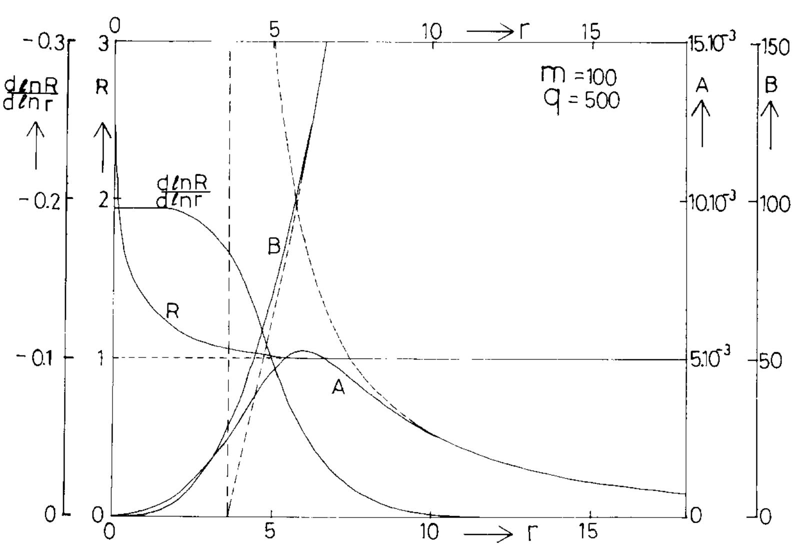

Fig. 1. The solution for $m = 100$ and $q = 500$. The qualitative features which hold for all solutions with $q > 0$ are that: (i) $A(0) = 0$ (horizon), (ii) $B(0) = 0$ (singularity) and (iii) $R(0) = \infty$.

cated. In the limit $q \downarrow 0$ the solution behaves rather strange: A and B vanish inside the horizon whereas $d \ln R/d \ln r$ becomes a step function located at the horizon. This limiting behaviour differs essentially from the other (exact) $q = 0$ solution namely the Freund–Rubin–Schwarzschild solution where $R \equiv 1$ for all r and A and B as in the usual Schwarzschild solution. Note that for $m = 0$ this strange behaviour is absent (except *at* the origin itself). Furthermore all these properties are also valid for the case with vanishing physical cosmological constant in particular near $r = 0$ the solution behaves qualitatively as in fig. 1. It is also worth noting that the solution is very different from the usual Reissner–Nordström solution with a charge q [5]. It is well known that this solution has two horizons for $q < m$ which "annihilate" at $q = m$. For $q > m$ the Reissner–Nordström solution exhibits a naked timelike singularity and we will point out next that this is different from the case we are discussing. Let us consider our solution for any $q > 0$, it has no singularity or horizon for a finite value of r. For $r = 0$ we have however both a horizon ($B = 0$) *and* a singularity ($A = 0$). One could say that we are dealing with a "tightly dressed" singularity. In view of eq. (5) this is possible *exactly* because the radius of the compact space tends to infinity for $r \to 0$.

[At this point it is worth observing that R starts deviating from its asymptotic value at the location of the would be horizon ($r \simeq m$), which can be much larger than R_∞ (i.e. the Planck scale).]

In order to analyse the causal structure of the solution at the origin we make an expansion around the origin and introduce some suitable coordinate system [5] (a la Kruskal or Penrose). The numerical results show that the quantity $d \ln R/ d \ln r$ tends to a constant:

$$-d \ln R/d \ln r|_{r=0} = \rho < \tfrac{2}{7} \tag{14}$$

(for the value $\rho = 2/7$, the denominator in the right-hand side of (5) would vanish, this is what happens if $q \downarrow 0$, leading to the peculiar behaviour pointed out before). From (7) we find that A tends to zero and from (5)–(6) also B has to go to zero and one easily finds:

$$A = \alpha r^a + O(r^{2a}), \quad B = \beta r^b + O(r^{2b}), \tag{15}$$

with

$$\tfrac{1}{2}(a-b) = 1 - 7\rho, \quad \tfrac{1}{2}(a+b) = 7(\rho+1)\rho/(2-7\rho). \tag{16}$$

The values of ρ, α and β are of course determined by the parameters m and q. Fitting the numerical solution with $m = 100$ and $q = 500$ for small r to (15) we found: $\rho = 0.1955$, $a = 2.222$, $b = 2.959$, $\alpha = 0.1317$ and $\beta = 0.6144$. These results are in excellent agreement with eq. (16).

Focussing on the t–r plane we first introduce conformally flat coordinates:

$$\bar{r} = 7(\beta/\alpha)^{1/2} \rho \int (B/A)^{1/2}\, dr \underset{r\to 0}{=} -r^{-7\rho},$$

$$\bar{t} = (\beta/\alpha)^{1/2}\, 7\rho t, \tag{17}$$

which brings the line-element in the form:

$$ds^2 = [-\alpha/\beta(7\rho)^2]\, B(-d\bar{t}^2 + d\bar{r}^2). \tag{18}$$

Since we are essentially only interested in the behaviour for $r \to 0$ we restrict ourselves (by rescaling s) to the metric:

$$d\bar{s}^2 = -r^b(-d\bar{t}^2 + d\bar{r}^2), \quad \bar{r} = -r^{-7\rho}. \tag{19}$$

To obtain a clear picture of what happens at the origin we should find coordinates for which the factor in front in (19) does not vanish for $r \to 0$. As usual we first go to advanced and retarded null-coordinates: $v = \bar{t} + \bar{r}$ and $w = \bar{t} - \bar{r}$ and subsequently define the Kruskal coordinates $\bar{v} = \exp(\bar{T}v)$ and $\bar{w} = -\exp(-\bar{T}w)$, where $\bar{T}$ is a constant. The metric (19) now becomes:

$$d\bar{s}^2 = r^b \exp(2\bar{T}r^{-7\rho})\, d\bar{v}\, d\bar{w}. \tag{20}$$

Alternatively one may think in terms of new timelike and spacelike coordinates $\bar{\tau} = \frac{1}{2}(\bar{v} + \bar{w})$ and $\bar{x} = \frac{1}{2}(\bar{v} - \bar{w})$ where $d\bar{v}\, d\bar{w} = d\bar{x}^2 - d\bar{\tau}^2$. The origin corresponds to the null-lines $\bar{\tau} \pm \bar{x} = 0$, because $\bar{\tau}^2 - \bar{x}^2 = -\exp(v - w) = -\exp(-2\bar{T}r^{-7\rho})$. Finally defining the Penrose coordinates $\hat{v} = \mathrm{arctg}\,(\bar{v})$ and $\hat{w} = \mathrm{arctg}\,(\bar{w})$ we arrive at the Penrose diagram as drawn in fig. 2. From this picture it is clear that the space–time we are considering does not have any causal inconsistencies. The singularity at $r = 0$ corresponds to a lightlike singularity, a novelty which nicely complements the well-known Schwarzschild solution with a spacelike singularity, and Reissner–Nordström solution with a timelike singularity. The coincidence of the horizon with the singularity has another important consequence, namely that it is not possible to analytically

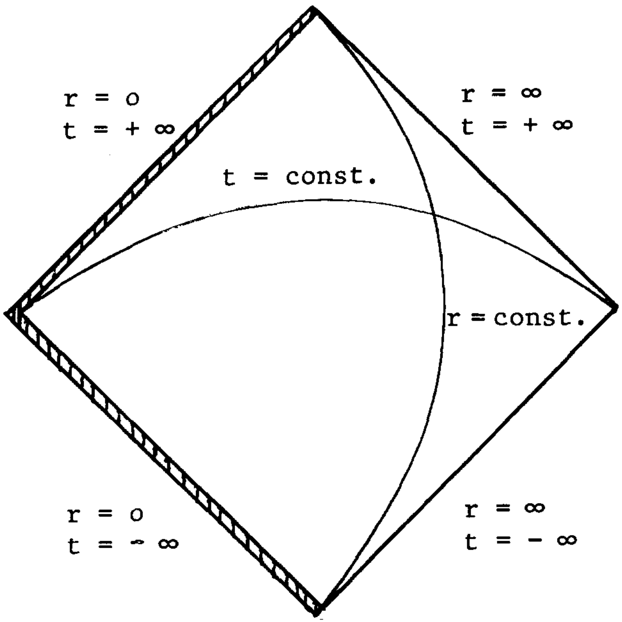

Fig. 2. The Penrose diagram for the solution. The singularity at $r = 0$ is lightlike and coincides with the horizon.

continue the coordinates beyond $r = 0$; there is no other universe connected to the singularity.

We now turn briefly to the question of the Hawking temperature [6], which is related to the occurrence of an horizon. To determine this temperature it is crucial that a coordinate system exists which is well behaved over the horizon. The easiest way to determine what the temperature would be, is to continue these coordinates to imaginary time. The periode then equals the inverse temperature (see ref. [7] however).

For the Schwarzschild and Reissner–Nordström solution the constant $\overline{T}$ in the transformation to Kruskal coordinates is uniquely fixed by demanding the line element to be regular across the horizon. In our case there is no choice for the value of $\overline{T}$ for which this can be achieved, leaving the Hawking temperature undetermined. One might argue that this turns the solution in a "quantummechanically naked" singularity. In the conventional solutions the distance of the singularity to the horizon is much larger than the Planck length. This justifies the, as far as gravity concerns, (semi)classical calculation of the Hawking effect. For a better understanding of the case we are discussing where horizon and singularity coincide, knowledge about quantum gravity is indispensable.

Let us finally come back to the interpretation of q as a charge. Due to nonlinearities the notion of charge cannot be defined through flux conservation, however it can always be done through its coupling to external test-particles. Following Kaluza and Klein [8] in their definition of the conventional charge Q, we will show that there is an interaction proportional to qQ^2. Let us first consider the case of global compactification, $q = 0$, $R \equiv 1$. Kaluza and Klein [8,9] introduce electromagnetic fields as off-diagonal elements in the metric. This reproduces correctly the Maxwell equations and the coupling to gravity. In order to reproduce the Lorentz force which a charged test-particle feels, one simply chooses the angular momentum in the extra dimension(s), which is a constant of the motion, proportional to the charge Q. Without loss of generality we can restrict the particle to the equators of S_2 and S_7, and the equation of motion becomes:

$$(\mathrm{d}r/\mathrm{d}s)^2 = \hat{\epsilon}^2/AB - (1 + \hat{k}^2/R^2)/A - \hat{l}^2/Ar^2 \,, \qquad (21)$$

where the constants of the motion are:

$$\hat{\epsilon} = B\,\mathrm{d}t/\mathrm{d}s\,, \quad \hat{l} = r^2\,\mathrm{d}\phi/\mathrm{d}s\,, \quad \hat{k} = R^2\,\mathrm{d}\psi_7/\mathrm{d}s\,. \qquad (22)$$

Since $\hat{k}^2/R^2$ is constant we can define the proper time to be $\tau = s(1 + \hat{k}^2/R^2)^{1/2}$ and we find:

$$(\mathrm{d}r/\mathrm{d}\tau)^2 = \epsilon^2/AB - 1/A - l^2/Ar^2 \,, \qquad (23)$$

with of course:

$$\epsilon = B\,\mathrm{d}t/\mathrm{d}\tau \quad \text{and} \quad l = r^2\,\mathrm{d}\phi/\mathrm{d}\tau\,. \qquad (24)$$

This amounts to the same result as the Kaluza–Klein approach where the line element is restricted to four dimensional space–time (if no electromagnetic fields are present).

We did this trivial exercise in order to deal with the case of non-constant R where the extra seven dimensions are treated as real physical degrees of freedom. In the case of local compactification $\hat{k}$ is still constant and can therefore be identified with $\delta \cdot Q$ (δ some constant). Therefore the equation of motion becomes ($R_\infty = 1$):

$$(\mathrm{d}r/\mathrm{d}\tau)^2 = \epsilon^2/AB - 1/A - l^2/Ar^2 + (\delta^2 Q^2/A)(1 - 1/R^2)\,, \qquad (25)$$

δ is some constant, which might depend on Q^2 through the scaling of the proper time. At the present level of discussion this is irrelevant. Substituting the asymptotic expressions (8) or (11) for R in (25) gives in the classical domain the following change in the effective potential:

$$\Delta V = -(2\delta^2 Q^2 q/A)\, \partial R/\partial q\,|_{q=0}\,,$$

which is attractive (for $q > 0$), independent of the sign of the charge of the test-particle and proportional to the "charge" q of the black hole. Since no explicit electromagnetic field is present gravitational and electromagnetic interactions get mixed up in the case of local compactification. This is unification in the true sense of the word.

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