ON THE ALGEBRAIC CHARACTERIZATION OF WITTEN'S TOPOLOGICAL YANG–MILLS THEORY

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We interpret in terms of "basic" cohomology the recently proposed supersymmetric, supergauge invariant formulation of topological Yang–Mills theory. Our interpretation shows that this formulation leads to the correct observables.

1. Introduction

In a recent series of articles [1,2], Witten investigates the expression of various topological invariants in terms of local field theory. The first examples of this sort we know of are due to Schwarz [3] who gives a field theory expression for the Ray–Singer analytic torsion [4], and are related to the quantization of differential forms [5]. The situations considered by Witten are of a more exotic type and lead to essentially non-linear theories, to be treated in the weak coupling regime. In principle, to obtain the sort of results one expects, a rigorous treatment of the renormalized perturbation expansion ought to be sufficient for a rigorous mathematical construction. Here, we shall be concerned with gauge fields and the recently discovered Donaldson invariants [6].

The following construction owes much to seminars by Singer, Baulieu [7] and Braam [8]. However, since local field theory is to be used [9], we find it necessary to characterize the model by a complete set of Ward identies. We believe that ref. [7], as well as subsequent proposals [10] are incomplete in this respect. The solution is to be found in an article by Horne [11]. The purpose of this note is to explain why, in more geometrical terms.

2. The differential algebra

As suggested in Witten's paper [1] (eq. (2.41)) and emphasized in ref. [7], one wishes to gauge-fix a topological invariant, e.g.

$$S_{\mathrm{inv}}(A)=\int_{M}\mathrm{tr}(F(A)\wedge F(A))\,, \tag{1}$$

where $F((A)$ is the curvature of a connection A on a principal G-bundle P(M, G), over a compact 4-manifold M, without boundary, and tr is an invariant polynomial over Lie G. The group G is assumed to be compact.

The action S_{inv} is, by essence, invariant under arbitrary variations of A:

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$$\delta A=\psi . \quad (2)$$

From now on, all fields are differential forms on M, taking values in ad(Lie G). One insists on gauge-fixing S_{inv}, leaving the gauge freedom pending till the end (because of the known Gribov difficulty), localizing the system on the self-dual connections. The corresponding gauge fixing term is

$$S^{(1)}=\int_M \operatorname{tr}(b\wedge b-b\wedge F^- -\bar\psi\wedge(\mathrm{D}\psi)^-) , \quad (3)$$

where F^- is the antiself-dual part of $F(A)$ – for some metric g on M –, b and $\bar\psi$ are antiself-dual two-forms and $(\mathrm{D}\psi)^-$ is the antiself-dual part of $\mathrm{D}\psi$, the covariant differential of the one-form ψ.

The new action $S_{\text{inv}}+S^{(1)}$ is invariant under the Slavnov symmetry:

$$s_1 A=\psi, \quad s_1\psi=0, \quad s_1\bar\psi=b, \quad s_1 b=0 \quad (4)$$

and satisfies

$$S^{(1)}=s_1\int_M \operatorname{tr}(\bar\psi\wedge(b-F^-)) . \quad (5)$$

It is still gauge invariant, i.e. invariant under

$$\delta A=\mathrm{D}\omega, \quad \delta\psi=[\psi,\omega] , \quad (6)$$

$$\delta\bar\psi=-[\omega,\bar\psi], \quad \delta b=[b,\omega] , \quad (7)$$

where $\omega\in$ Lie $\mathscr{G}$, $\mathscr{G}$ being the gauge group of P(M, G). This yields the nilpotent s_2 operation:

$$s_2 A=\psi-\mathrm{D}\omega, \quad s_2\psi=[\psi,\omega], \quad s_2\bar\psi=-[\omega,\bar\psi]+b,$$

$$s_2 b=[b,\omega], \quad s_2\omega=-\tfrac{1}{2}[\omega,\omega] . \quad (8)$$

The action $S_{\text{inv}}+S^{(1)}$ is invariant under s_2 and does not depend on ω (the Faddeev–Popov ghost for Lie $\mathscr{G}$). Except for the $\int b\wedge b$ term, it is invariant under (φ is odd)

$$\delta b=[\bar\psi,\varphi], \quad \delta\psi=\mathrm{D}\varphi , \quad (9)$$

which yields the nilpotent operation s (now with φ even):

$$sA=\psi-\mathrm{D}\omega, \quad s\psi=[\psi,\omega]-\mathrm{D}\varphi ,$$

$$s\bar\psi=-[\omega,\bar\psi]+b, \quad sb=[b,\omega]-[\bar\psi,\varphi] ,$$

$$s\varphi=[\varphi,\omega], \quad s\omega=-\tfrac{1}{2}[\omega,\omega]+\varphi . \quad (10)$$

It is easy to modify $S^{(1)}$ in such a way that it is invariant under (10). Following eq. (5) we find

$$\hat S^{(1)}=s\int_M \operatorname{tr}(\bar\psi\wedge(b-F^-))$$

$$=\int_M \operatorname{tr}(b\wedge b-b\wedge F^- -\bar\psi\wedge(\mathrm{D}\psi)^- +\bar\psi\wedge[\bar\psi,\varphi]) . \quad (11)$$

Notice that in eq. (10), sb needs a φ dependent term in order for s to be nilpotent. The φ invariance can be gauge fixed in a gauge invariant way using the gauge function $\mathrm{D}^*\psi$:

$$\hat S^{(2)}=\int_M \operatorname{tr}({}^*\beta\wedge \mathrm{D}^*\psi+{}^*\bar\varphi\wedge(\mathrm{D}^*\mathrm{D}\varphi+[{}^*\psi,\psi])) . \quad (12)$$

Including $\hat S^{(2)}$ in the action, one gets the Slavnov symmetry defined by (10), together with

$$s\bar\varphi=\beta+[\bar\varphi,\omega], \quad s\beta=-[\omega,\beta]+[\varphi,\bar\varphi] , \quad (13)$$

such that we have the following expression:

$$\hat S^{(1)}+\hat S^{(2)}=s\int_M \operatorname{tr}({}^*\bar\varphi\wedge \mathrm{D}^*\psi+\bar\psi\wedge(b-F^-)) . \quad (14)$$

A few remarks are in order:

(i) $S_{\text{inv}}+\hat S^{(1)}+\hat S^{(2)}$ is not quite the most general ω independent, gauge invariant, renormalizable action invariant under s – actually of the form $S_{\text{inv}}+sS_{\text{gf}}$: one may add an extra term compatible with ghost number neutrality and renormalizability as in refs. [1,2,10,11] and of the form $s\int \operatorname{tr}(\beta[\varphi,\bar\varphi])$. Both ω independence and gauge invariance are essential.

(ii) Changing generators according to

$$\psi'=\psi-\mathrm{D}\omega, \quad b'=b-[\omega,\bar\psi] ,$$

$$\varphi'=\varphi-\tfrac{1}{2}[\omega,\omega], \quad \beta'=\beta+[\bar\varphi,\omega] , \quad (15)$$

the s-operation assumes the form

$$sA=\psi', \quad s\psi'=0 , \quad s\bar\psi=b', \quad sb'=0 ,$$

$$s\omega=\varphi', \quad s\varphi'=0 , \quad s\bar\varphi=\beta', \quad s\beta'=0 . \quad (16)$$

It therefore has vanishing cohomology as well as vanishing cohomology mod d. The desired cohomology [1,2] is, however, not the local cohomology of s mod d, but its restriction to ω independent, gauge invariant objects. What is involved is equivariant cohomology [8] [#1], or rather its original form, namely "basic" cohomology [13], which is exactly adapted to the present local field theory context, as we shall see.

(iii) It is interesting to observe that s can be split into a sum of two anticommuting differentials and that the algebra can be cast in a supersymmetric form, which is *distinct* from that of ref. [11], if we insist that, as we will demonstrate, s generates the supersymmetry. However, the superfield content will be that of ref. [11] (without imposing the gauge condition $\omega=0$). Details are given in section 4.

3. The "basic" cohomology of s

The differential algebra defined by the structure eqs. (10) and (13) has the following property, which makes it a differential algebra with an action of the gauge Lie algebra; for $\lambda\in\mathrm{Lie}\,\mathscr{G}$ define

$$\delta_\lambda\psi=[\psi,\lambda],\quad \delta_\lambda A=\mathrm{D}\lambda\,,$$
$$\delta_\lambda b=[b,\lambda],\quad \delta_\lambda\bar\psi=-[\lambda,\bar\psi]\,,$$
$$\delta_\lambda\varphi=[\varphi,\lambda],\quad \delta_\lambda\omega=-[\lambda,\omega]\,,$$
$$\delta_\lambda\bar\varphi=[\bar\varphi,\lambda],\quad \delta_\lambda\beta=-[\lambda,\beta]\,. \tag{17}$$

Define also for $\lambda\in\mathrm{Lie}\,\mathscr{G}$, ι_λ by

$$\iota_\lambda A=\iota_\lambda\psi=\iota_\lambda\bar\psi=\iota_\lambda b=\iota_\lambda\varphi=\iota_\lambda\bar\varphi=\iota_\lambda\beta=0\,,$$
$$\iota_\lambda\omega=\lambda\,. \tag{18}$$

One can easily check that

$$\delta_\lambda=\iota_\lambda s+s\iota_\lambda \tag{19}$$

and one has the classical [13] commutation rules

$$[\iota_\lambda,\iota_\mu]_+=0,\quad [\delta_\lambda,\delta_\mu]_-=\delta_{[\lambda,\mu]},\quad [\delta_\lambda,\iota_\mu]_-=\iota_{[\lambda,\mu]},$$
$$[s,\iota_\lambda]_+=\delta_\lambda,\quad [s,\delta_\lambda]_-=0\,. \tag{20}$$

This makes $\{\delta_\lambda,\iota_\mu|\lambda,\mu\in\mathrm{Lie}\,\mathscr{G}\}$ into a graded Lie algebra. Recall that $S_{\mathrm{tot}}=S_{\mathrm{inv}}+\hat S^{(1)}+\hat S^{(2)}$ fulfils

$$sS_{\mathrm{tot}}=\delta_\lambda S_{\mathrm{tot}}=\iota_\mu S_{\mathrm{tot}}=0,\quad \lambda,\mu\in\mathrm{Lie}\,\mathscr{G}\,. \tag{21}$$

In technical terms S_{tot} is a "basic" [13] local functional for the differential structure (10), (13), with the Lie $\mathscr{G}$ action defined by (17), (18).

[#1] This is an amplification of a remark by Braam (see ref. [8]). See also refs. [7,12].

Now let us turn λ and μ into ghosts, in the usual fashion (λ odd and μ even) and define

$$W=\delta+\iota\,, \tag{22}$$

where δ and ι are obtained by (17), (18) on all fields except for A, ω, λ and μ for which

$$WA=-\mathrm{D}\lambda,\quad W\omega=-[\lambda,\omega]-\mu\,,$$
$$W\lambda=-\tfrac12[\lambda,\lambda],\quad W\mu=[\mu,\lambda]\,. \tag{23}$$

One easily shows that $W^2=0$. Adjoining λ and μ as new generators to our differential algebra, we still have a choice to define $s\lambda$ and $s\mu$. In particular, if we define s on λ and μ by

$$s\lambda=\mu,\quad s\mu=0\,, \tag{24}$$

we obtain

$$[s,W]_+=0\,. \tag{25}$$

The comparison with the supersymmetric formalism of ref. [11] is now straightforward. In terms of the primed variables defined in eq. (15), one may introduce the superfields

$$\mathscr{A}_x=A+\theta\psi',\quad \mathscr{A}_\theta=\omega+\theta\varphi'\,,$$
$$\Psi=\bar\psi+\theta b',\quad \Phi=\bar\varphi+\theta\beta'\,. \tag{26}$$

Then one has

$$s=\partial/\partial\theta\,. \tag{27}$$

The supergauge transformation ghost

$$\Lambda=\lambda+\theta\mu\,, \tag{28}$$

fulfils $W\Lambda=-\frac12[\Lambda,\Lambda]$ and s still acts on Λ by $\partial/\partial\theta$. W acts on all fields by supergauge transformations, with $\mathscr{A}_x$, $\mathscr{A}_\theta$ a superconnection and Ψ, Φ transforming under the adjoint representation.

In terms of the unprimed variables the action and local cohomology mod d are characterized by ω independence and gauge invariance, as we have already remarked. In terms of the primed variables and the supersymmetric formulation of ref. [11], this is equivalent to supersymmetry (invariance under $\partial/\partial\theta$) and supergauge invariance. So, this equivalence proves in particular that the supersymmetric supergauge invariant cohomology is identified with the "basic" cohomology, which is known to be correct [1,2,7,8]. We refer to ref. [11] for the s-invariant gauge fixing of W.

4. An alternative supersymmetry

In this section we discuss the alternative supersymmetry mentioned at the end of section 2. One may split s in eqs. (10) and (13) as

$$s=\sigma+w\,, \tag{29}$$

with

$$\sigma A=\psi,\quad \sigma\psi=0,\quad \sigma\bar\psi=b,\quad \sigma b=0\,,$$
$$\sigma\omega=\varphi,\quad \sigma\varphi=0,\quad \sigma\bar\varphi=\beta,\quad \sigma\beta=0 \tag{30}$$

and

$$w\psi=[\psi,\omega]-\mathrm{D}\varphi,\quad wA=-\mathrm{D}\omega\,,$$
$$wb=[b,\omega]-[\bar\psi,\varphi],\quad w\bar\psi=-[\omega,\bar\psi]\,,$$
$$w\varphi=[\varphi,\omega],\quad w\omega=-\tfrac{1}{2}[\omega,\omega]\,,$$
$$w\bar\varphi=[\bar\varphi,\omega],\quad w\beta=-[\omega,\beta]+[\varphi,\bar\varphi]\,. \tag{31}$$

One can easily check that $\sigma^2=w^2=[\sigma,w]_+=0$. This structure suggests the use of a supersymmetric formalism. Let

$$\boldsymbol{A}=A+\theta\psi,\quad \boldsymbol{\Omega}=\omega+\theta\varphi\,,$$
$$\bar{\boldsymbol{\Psi}}=\bar\psi+\theta b,\quad \bar{\boldsymbol{\Phi}}=\bar\varphi+\theta\beta\,. \tag{32}$$

Then, in terms of the superfields:

$$\sigma=\partial/\partial\theta\,, \tag{33}$$

and w is a supergauge transformation:

$$w\boldsymbol{A}=-\mathbf{D}(\boldsymbol{A})\boldsymbol{\Omega},\quad w\boldsymbol{\Omega}=-\tfrac{1}{2}[\boldsymbol{\Omega},\boldsymbol{\Omega}]\,,$$
$$w\bar{\boldsymbol{\Psi}}=-[\boldsymbol{\Omega},\bar{\boldsymbol{\Psi}}],\quad w\bar{\boldsymbol{\Phi}}=[\bar{\boldsymbol{\Phi}},\boldsymbol{\Omega}]\,, \tag{34}$$

where the covariant differential $\mathbf{D}(\boldsymbol{A})$ is given by

$$\mathbf{D}(\boldsymbol{A})\boldsymbol{\Omega}=\mathrm{d}\boldsymbol{\Omega}+[\boldsymbol{A},\boldsymbol{\Omega}]\,. \tag{35}$$

Eq. (34) defines a differential superalgebra with a super Lie algebra action in terms of

$$\Lambda=\lambda+\theta\mu,\quad \lambda,\,\mu\in\mathrm{Lie}\,\mathscr{G}\,. \tag{36}$$

We define δ_Λ according to

$$\delta_\Lambda\boldsymbol{A}=\mathbf{D}(\boldsymbol{A})\Lambda,\quad \delta_\Lambda\boldsymbol{\Omega}=[\boldsymbol{\Omega},\Lambda]\,,$$
$$\delta_\Lambda=[\bar{\boldsymbol{\Psi}},\Lambda],\quad \delta_\Lambda=[\bar{\boldsymbol{\Phi}},\Lambda] \tag{37}$$

and ι_Λ according to

$$\iota_\Lambda\boldsymbol{A}=\iota_\Lambda\bar{\boldsymbol{\Psi}}=\iota_\Lambda\bar{\boldsymbol{\Phi}}=0,\quad \iota_\Lambda\boldsymbol{\Omega}=\Lambda\,. \tag{38}$$

Then we have

$$\delta_\Lambda=[w,\iota_\Lambda]\,. \tag{39}$$

5. Concluding remarks

The algebraic set-up proposed in ref. [11] has been shown to describe the "basic" cohomology adapted to the characterization of a perturbative treatment [9] of the situation described by Witten [1,2] in terms of equivariant cohomology [8] (see also footnote 1). There are two heavy technical problems to be dealt with:

(i) Perturbative renormalization theory for a field theory associated with an arbitrary compact manifold without boundary in a particular topological sector.

(ii) The proper treatment of different vacua and the inclusion in the s–W operation of global zero modes, that ought to make the theory not completely empty.

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