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PHYSICS LETTERS B

Physics Letters B 552 (2002) 214–222

www.elsevier.com/locate/npe

Defining $\langle A^2 \rangle$ in the finite volume Hamiltonian formalism

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Received 22 October 2002; received in revised form 6 November 2002; accepted 4 December 2002

Editor: P.V. Landshoff

Abstract

It is shown how in principle for non-Abelian gauge theories it is possible in the finite volume Hamiltonian framework to make sense of calculating the expectation value of $\|A\|^2 = \int d^3x (A_i^a(\vec{x}))^2$. Gauge invariance requires one to replace $\|A\|^2$ by its minimum over the gauge orbit, which makes it a highly non-local quantity. We comment on the difficulty of finding an expression for $\|A\|_{\min}^2$ analogous to that found for the Abelian case, and the relation of this question to Gribov copies. We deal with these issues by implementing the Hamiltonian on the so-called fundamental domain, with appropriate boundary conditions in field space, essential to correctly represent the physics of the problem.

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1. Introduction

The question of a dimension-two condensate in QCD or in pure gauge theories [1], is a fascinating one. Condensates are believed to be connected to the non-perturbative structure of the theory. In a theory without manifest dimensional parameters the only evident operator for such a condensate is in terms of the vector potential itself $V^{-1}\|A\|^2$. This is certainly the simplest object one can imagine in a pure gauge theory, but it suffers from the obvious defect that it is not gauge invariant and so cannot have any physical meaning. In perturbation theory this can be remedied when one replaces the vector potential by its transverse part, satisfying $\partial_\mu A^\mu(x) = 0$.

Irrespective of the question if one could separate off the perturbative contribution, it is not at all clear

if one can even make sense of calculating $\langle A^2 \rangle$ non-perturbatively. It is only the latter point we address in this Letter, by using the Hamiltonian framework. An essential ingredient is to consider $\|A\|_{\min}^2$, which minimizes $\|A\|^2$ along the gauge orbit. In the Abelian case this can be expressed in terms of the field strength [2], with some non-locality involved. In Section 2 we point out that the difficulty to find a similar formula for $\|A\|_{\min}^2$ in the non-Abelian case is related to the Gribov copy problem [3].

We will then show in Sections 3 and 4 how the Hamiltonian formulation in a finite volume, using the Coulomb gauge [4], can deal with the Gribov issue by restricting the transverse gauge fields to the fundamental domain [5,6]. On its boundary, $\|A\|_{\min}^2$ is degenerate [7], and it is this that allows one to consistently define $\langle A^2 \rangle = \langle 0|V^{-1}\|A\|_{\min}^2|0 \rangle$. Non-perturbative contributions typically arise when the wave functional starts to reach the boundary of the fundamental domain. This has been successfully implemented in the

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past to calculate the low-lying spectrum in a finite volume (for a review see Ref. [8]).

The finite volume cutoff allows us to define the contribution coming from the low-lying modes for which the wave functional is affected by the boundary of the fundamental domain, in terms of an effective Hamiltonian for these modes. By excluding the perturbative contributions coming from the modes that are integrated out, in as far as they do *not interact* with the low-lying modes kept in the effective Hamiltonian, this gives by construction a finite result. This is presented in Section 5, and illustrates that $\langle A^2 \rangle$, indeed, receives a non-perturbative contribution due to the boundary conditions in field space, properly reflecting the non-trivial geometry of the configuration space [9].

We resist the temptation of subtracting the perturbative result obtained with the effective Hamiltonian, to avoid the usual difficulties with defining a condensate unambiguously. The aim of this Letter is to demonstrate that there is a way to define $\langle A^2 \rangle$ beyond perturbation theory.

2. Abelian gauge theories

An elegant expression for Abelian gauge theories exists, that splits $\|A\|^2$ in transverse and longitudinal parts, which when expressed in momentum space (the Fourier components are denoted by a tilde)

$$\|A\|^2 = \int d^n p \frac{\tilde{F}^{\mu\nu}(p) \tilde{F}_{\mu\nu}^*(p)}{p_\mu p^\mu} + \int d^n p \frac{p^\mu \tilde{A}_\mu(p) p^\nu \tilde{A}_\nu^*(p)}{p_\mu p^\mu}, \quad (1)$$

holds in any dimension [2]. It relies on the well-known vector identity

$$\frac{1}{2} \tilde{F}^{\mu\nu}(p) \tilde{F}_{\mu\nu}^*(p) = p_\mu p^\mu \tilde{A}^\nu(p) \tilde{A}_\nu^*(p) - p^\mu p^\nu \tilde{A}_\mu(p) \tilde{A}_\nu^*(p), \quad (2)$$

useful in setting up perturbation theory. Since minimizing $\|A\|^2$ along the gauge orbit implies the gauge field at this minimum satisfies the gauge condition $\partial^\mu A_\mu(x) = 0$, or $p^\mu \tilde{A}_\mu(p) = 0$, this implies we have

a gauge invariant expression for $\|A\|_{\min}^2$,

$$\|A\|_{\min}^2 = \int d^n p \frac{\tilde{F}_{\mu\nu}^2(p)}{p_\mu p^\mu}, \quad (3)$$

with only a limited amount of non-locality.

There is a small problem to address here because p^2 cannot be inverted for zero-momentum. In coordinate space this could give rise to boundary terms [2]. The problem persists with periodic boundary conditions also. This is interesting in that it reveals a subtle issue related to Gribov copies [3], which are essential to the non-Abelian problem.

With boundary conditions periodic in a length L , the integral over momenta is replaced by a sum. The zero-momentum components of $F_{\mu\nu}$ and $\partial_\mu A^\mu$ vanish, but not those for A_μ . This means that we have to replace $\|A\|_{\min}^2$ by $\|A\|_{\min}^2 - \tilde{A}_\mu^*(0) \tilde{A}^\mu(0) = \|A\|_{\min}^2 - L^{-n} (\int d^n x A_\mu(x))^2$. This may seem an insignificant modification as $L \rightarrow \infty$, but it is exactly what is needed to deal with the problem of Gribov copies: different gauge fields related by a gauge transformation that satisfy the gauge condition $\partial_\mu A^\mu(x) = 0$, but for which the value of $\|A\|^2$ differ.

These Gribov copies can even be present in the Abelian theory (for finite volume and periodic boundary conditions). However, the difference in A between Gribov copies in this case has zero momentum. After all, with $[h]A_\mu(x)$ the gauge field obtained by a gauge transformation $h(x)$ from $A_\mu(x)$, requiring $\partial_\mu([h]A^\mu(x)) = \partial_\mu A^\mu(x) = 0$ implies $\partial_\mu^2 \log h(x) = 0$. This fixes the allowed gauge transformations to be of the form $h(x) = \exp(2\pi i x^\mu n_\mu / L)$, with n_μ integer (as imposed by the periodic boundary conditions). It means that the difference in $\|A\|^2$ between different Gribov copies is only in the zero-momentum component and we have

$$\begin{aligned} \|A\|^2 - L^{-n} \left(\int d^n x A_\mu(x) \right)^2 \\ = \sum_{p \neq 0} \frac{\tilde{F}^{\mu\nu}(p) \tilde{F}_{\mu\nu}^*(p)}{p_\mu p^\mu} + \sum_{p \neq 0} \frac{p^\mu \tilde{A}_\mu(p) p^\nu \tilde{A}_\nu^*(p)}{p_\mu p^\mu}, \end{aligned} \quad (4)$$

and

$$\|A\|_{\min}^2 - L^{-n} \left(\int d^n x A_\mu(x) \right)^2$$

$$= \sum_{p \neq 0} \frac{\tilde{F}^{\mu\nu}(p) \tilde{F}_{\mu\nu}^*(p)}{p_\mu p^\mu}. \quad (5)$$

This formula is also correct when instead of taking the absolute minimum we have a stationary point. It is this that causes the Gribov problem: there are many stationary points of $\|A\|^2$ along a given gauge orbit, where at each of these stationary points the (Coulomb or Landau) gauge condition $\partial_\mu A^\mu(x) = 0$ holds. For Abelian gauge theories, this problem only occurs in a finite volume with periodic boundary conditions and the Gribov copies can be fully classified in terms of the zero-momentum component of the gauge field.

It is amusing to observe, since the vector identity is still true, that in the non-Abelian theory (in a finite volume with periodic boundary conditions) we may again write

$$\begin{aligned} \|A\|^2 - L^{-n} \left(\int d^n x A_\mu^a(x) \right)^2 \\ = \sum_{p \neq 0} \frac{\tilde{f}^{a\mu\nu}(p) \tilde{f}_{\mu\nu}^{a*}(p)}{p_\mu p^\mu} + \sum_{p \neq 0} \frac{p^\mu \tilde{A}_\mu^a(p) p^\nu \tilde{A}_\nu^{a*}(p)}{p_\mu p^\mu}, \end{aligned} \quad (6)$$

with $f_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x)$ and

$$\begin{aligned} \|A\|_{\min}^2 - L^{-n} \left(\int d^n x A_\mu^a(x) \right)^2 \\ = \sum_{p \neq 0} \frac{\tilde{f}^{a\mu\nu}(p) \tilde{f}_{\mu\nu}^{a*}(p)}{p_\mu p^\mu}. \end{aligned} \quad (7)$$

This reveals at once the problem for the non-Abelian case: the right-hand side involving only the “curl” part of the field tensor is not gauge invariant, and the problem of Gribov copies cannot be restricted to the zero-momentum component of the gauge field. Indeed, explicit examples are known that illustrate this point [7]. Minimizing along the gauge orbit has the complexity of a spin glass problem, with many local minima, which from the computational point of view makes it in practice impossible to identify the absolute minimum.

3. Non-Abelian gauge theories

We have seen that there appears to be no simple gauge invariant expression for $\|A\|_{\min}^2$ in non-Abelian

gauge theories, even allowing for non-locality. Certainly a formula for $\|A\|^2$ similar to the Abelian case, where we have a gauge invariant expression plus something vanishing at the stationary points of $\|A\|^2$ cannot apply. This would give the same value for $\|A\|^2$ at Gribov copies, while we know on the contrary that generically $\|A\|^2$ is different for such copies.

Therefore, we would like to turn to the Hamiltonian picture of non-Abelian gauge theory to provide some insight into the question of $\|A\|_{\min}^2$. What we would like to find is that in a certain sense $\|A\|_{\min}^2$ viewed as a quantum-mechanical operator can have a non-trivial expectation value, as we shall now explain.

In the Hamiltonian picture [4], where $A_0 = 0$, one considers wave functionals on field space. The “coordinates” are the spatial components of the vector potential at every point in ordinary space \vec{x} , $A_i^a(\vec{x})$. In a lattice formulation, for example, there can be a finite number of coordinates. Or in momentum space one may use $\tilde{A}_i^a(\vec{k})$ as the variables. In any event we imagine a wave function in these coordinates $\Psi(A)$.

Now in the simplest case of a free Abelian theory, one has a problem equivalent to the ordinary harmonic oscillator for the modes in momentum space. Evidently, there is a non-zero value of $\|A\|^2$ for each mode, since this is just the spread of the wave function in the ground state of the oscillator. This is what one expects for the ordinary vacuum and we shall call this the “perturbative contribution”, since when summed over all modes it represents the well-known (infinite) zero-point motion of free fields.

When A becomes large Ψ will be sensitive to the A^3 and A^4 terms in the potential energy $\frac{1}{2} B_i^a(\vec{x})^2$, with $B_i^a(\vec{x})$ the Yang–Mills magnetic field. These correspond to the non-linearities of the theory and determine the directions in which $\Psi(A)$ can spread. When $\Psi(A)$ can no longer be neglected near the boundary of the fundamental domain, it is sensitive to the boundary conditions required to make the problem well defined. It is this that can no longer be described in perturbation theory, leading to non-perturbative contributions.

The implementation of the Hamiltonian approach depends very much on whether and how the gauge condition $\partial_i A^i(\vec{x}) = 0$ is handled. In principle, one may not apply any condition of this type at all and simply assume that the Ψ is constant in gauge directions, that is, constant over a gauge orbit. Although this is

conceptually simple and is usually the approach in lattice simulations, it is remote from standard perturbation theory and difficult to apply for concrete Hamiltonian calculations. In particular for our present question this would necessitate an explicit gauge invariant expression for an $\|A\|_{\min}^2$ operator, which as explained in the previous section, we do not have.

The more common approach in the Hamiltonian method is thus to formulate the problem in terms of one representative A field configuration on a gauge orbit in order to reduce the number of variables. This configuration is found by imposing $\partial_i A^i(\vec{x}) = 0$ and one then uses the Faddeev–Popov method to find the volume of the gauge orbit when integrating over A configurations [9]. However, this leads to the problem of Gribov copies, since in the non-Abelian case there is more than one A configuration with $\partial_i A^i(\vec{x}) = 0$ on a given gauge orbit. For the question of $\|A\|_{\min}^2$ the existence of Gribov copies means that $\partial_i A^i(\vec{x}) = 0$ no longer determines that configuration where $\|A\|^2$ is an absolute minimum.

A way of dealing with these complications is to define a fundamental domain [5] where there is only one A configuration with $\partial_i A^i(\vec{x}) = 0$ on each gauge orbit. Restricting the variables to this domain leads to a well-defined quantum-mechanical problem, where we may calculate the expectation value of A^2 . The price of this simplification, however, is a complicated topology in A space on the boundary of this domain, as we shall now explain. To keep things well defined, we introduce a finite volume in ordinary space as an infrared cutoff, like the torus T^3 or the sphere S^3 . For the torus, zero-momentum modes have to be treated carefully, but one has learned how to deal with this, see a recent review in Ref. [8].

The Hamiltonian formalism provides more intuition on how to deal with non-perturbative contributions in situations where semi-classical techniques can no longer be used. The high energy modes can be well-approximated by harmonic oscillator contributions to the wave functional. In the direction of these field modes the potential energy rises steeply. Their contributions, which include regulating the ultraviolet behavior, can presumably be treated perturbatively, in particular giving rise to the running of the coupling constant.

The finite volume allows us to have a well-defined mode expansion in momentum space. Due to the clas-

sical scale invariance, the Hamiltonian can be formulated in terms of dimensionless fields. This can be extended to the quantum theory, as Ward identities allow for a field definition without anomalous scaling. Thus we absorb the bare coupling constant in the gauge field. In these conventions the field strength is given, in terms $A_i(\vec{x}) = A_i^a(\vec{x})T^a$ (T^a the anti-Hermitian generators, normalized according to $\text{tr}(T^a T^b) = -\frac{1}{2}\delta_{ab}$), by

$$F_{ij}(\vec{x}) = F_{ij}^a(\vec{x})T^a \\ = \partial_i A_j(\vec{x}) - \partial_j A_i(\vec{x}) + [A_i(\vec{x}), A_j(\vec{x})] \quad (8)$$

and the Hamiltonian density reads

$$\mathcal{H}(\vec{x}) = -\frac{1}{2}g^2 \left(\frac{\partial}{\partial A_j^a(\vec{x})} \right)^2 + \frac{1}{2g^2} (B_j^a(\vec{x}))^2, \quad (9)$$

where $B_k^a(\vec{x}) = \frac{1}{2}\epsilon_{ijk}F_{ij}^a(\vec{x})$. When all fields and coordinates are expressed in units of L (with $Q_i^a(\vec{x})$ the usual expression for the gauge field, $A_i^a(\vec{x}) = gL Q_i^a(L\vec{x})$), apart from the overall scaling dimension of the Hamiltonian ($1/L$), only the running coupling introduces a non-trivial volume dependence [8].

Therefore, the only sensitivity to the length scale L is through an increasing coupling as we increase L . An increasing coupling will cause spreading of the wave functional, simply because the overall strength of the potential (proportional to $1/g^2$) is reduced. The essential additional ingredient required to address non-perturbative effects is the boundary conditions in *field space*, at the boundary of the fundamental domain. Only in this way can gauge invariance be implemented properly at all stages. On the other hand, asymptotic freedom guarantees that in small volumes the running coupling is small and it thus keeps the wave functional localized near the classical vacuum manifold. What has become clear [8] is that the transition from finite to infinite volume is driven by field fluctuations that cross the barrier which is associated with tunneling between different classical vacua. This is natural, since this barrier (the finite volume sphaleron, which will typically lie on the boundary of the fundamental domain), will be the direction beyond which the wave functional can first spread most significantly, as it provides the lowest mountain pass in the energy landscape.

4. The fundamental domain

We will summarize how to completely fix the gauge and show that the boundary of the fundamental domain, unlike its interior, has gauge copies that implement the non-trivial topology of field space. The essential observation that allows one to define $\|A\|_{\min}^2$ as a proper gauge invariant quantity is that this minimum along the gauge orbit is degenerate when the associated gauge fields (by definition related by a gauge transformation) represent points on the boundary of the fundamental domain that are to be identified.

Restricting to three space dimensions, we will now be a bit more precise about how to minimize the L^2 norm of the vector potential along the gauge orbit [5, 6] (recall that the vector potential $A_i(\vec{x})$ is anti-Hermitian).

$$\begin{aligned} \|[h]A\|^2 = & - \int_M \text{tr}((h^{-1}(\vec{x})A_i(\vec{x})h(\vec{x}) \\ & + h^{-1}(\vec{x})\partial_i h(\vec{x}))^2). \end{aligned} \quad (10)$$

The integral over the finite spatial volume M is with the appropriate canonical volume form. We introduce the short-hand notation $[h]A$ for a gauge transformation $h(\vec{x})$. Expanding around the minimum of Eq. (10), writing $h(\vec{x}) = \exp(X(\vec{x}))$ ($X(\vec{x})$ is, like the gauge field $A_i(\vec{x})$, an element of the Lie-algebra) one easily finds

$$\begin{aligned} \|[h]A\|^2 = & \|A\|^2 + 2 \int_M \text{tr}(X\partial_i A_i) + \int_M \text{tr}(X^\dagger FP(A)X) \\ & + \frac{1}{3} \int_M \text{tr}(X[[A_i, X], \partial_i X]) \\ & + \frac{1}{12} \int_M \text{tr}([\mathcal{D}_i X, X][\partial_i X, X]) + \mathcal{O}(X^5), \end{aligned} \quad (11)$$

where $FP(A)$ is Faddeev–Popov operator ($\text{ad}(A)X \equiv [A, X]$)

$$FP(A) = -\partial_i \mathcal{D}_i(A) \equiv -\partial_i (\partial_i + \text{ad}(A_i)). \quad (12)$$

At a local minimum the vector potential is therefore transverse, $\partial_i A_i = 0$, and $FP(A)$ must be a positive operator. The set of all these vector potentials is by

definition the Gribov region Ω . Using the fact that $FP(A)$ is linear in A , Ω is seen to be a convex subspace of the set of transverse gauge fields Γ . Its boundary $\partial\Omega$ is called the Gribov horizon. At the Gribov horizon, the *lowest* non-trivial eigenvalue of the Faddeev–Popov operator vanishes, and points on $\partial\Omega$ are associated with coordinate singularities. Any point on $\partial\Omega$ can be seen to have a finite distance to the origin of field space and in some cases even uniform bounds can be derived [10,11].

The Gribov region is the set of *local* minima of the norm functional, Eq. (10), and needs to be further restricted to the *absolute* minima to form the fundamental domain [5], which will be denoted by Λ . The fundamental domain is clearly contained within the Gribov region. To show that also Λ is convex, we define an operator $FP_f(A)$ via

$$\begin{aligned} \|[h]A\|^2 - \|A\|^2 = & \int_M \text{tr}(A_i^2) - \int_M \text{tr}((h^{-1}A_i h + h^{-1}\partial_i h)^2) \\ = & \int_M \text{tr}(h^{-1}FP_f(A)h), \\ FP_f(A) \equiv & -\partial_i (\partial_i + A_i), \end{aligned} \quad (13)$$

(remember that in our conventions $\text{tr}(A_i^2)$ is negative), where $FP_f(A)$ acts on Lie-group valued functions and is similar to the Faddeev–Popov operator (which acts on Lie-algebra valued functions). Both $FP(A)$ and $FP_f(A)$ are Hermitian operators when A is a stationary point of the norm functional, i.e., for A transverse. The fundamental domain Λ is the set of gauge fields A for which Eq. (13) has its minimum at zero when varying h over the gauge group. (If this minimum is unique it occurs for $h = 1$.) Using that $FP_f(A)$ is linear in A , the convexity of Λ is automatic: A line connecting two points in Λ lies within Λ .

If we would not specify anything further, since a convex space is contractible, the fundamental region could never reproduce the non-trivial topology of the field space. This means that Λ should have a boundary [7]. Indeed, as Λ is contained in Ω , this means Λ is also bounded in each direction. Consider a gauge orbit and two gauge configurations on it, giving the absolute and first relative minimum of $\|A\|^2$, respectively. In general the two configurations

are connected by a finite, or even a “big”, gauge transformation. Now take the “ray” that extends from the relative minimum configuration to $A = 0$. There it will have $\|A\|^2 = 0$. Its gauge copy, initially an absolute minimum, will also vary continuously as we go along the ray towards $A = 0$, but will not in general have $\|A\|^2 = 0$ at the end of the variation. Therefore the norms of each of these two configurations must pass each other during this variation. At the crossing we have degenerate minima of $\|A\|^2$ at distinct points on the gauge orbit. These correspond to different points of the boundary of Λ , identified by gauge equivalence. This gives the problem its non-trivial topology.

When L denotes the linear size of the spatial volume M , we may express the gauge fields in the dimensionless combination of LA (in our conventions the fields have no anomalous scale dependence), and *the shape and geometry of the Gribov and fundamental regions are scale independent*. We should note that the norm functional is degenerate along the constant gauge transformations and indeed the Coulomb gauge does not fix these gauge degrees of freedom. We simply demand that the wave functional is in the singlet representation under the constant gauge transformations. Therefore, with G the gauge group, Λ/G represents the gauge invariant configuration space, for which Λ is assumed to include the non-trivial boundary identifications that restore the non-trivial topology of this space.

If a degeneracy at the boundary is continuous, other than by constant gauge transformations, one necessarily has at least one non-trivial zero eigenvalue for $FP(A)$ and the Gribov horizon will touch the boundary of the fundamental domain at these so-called singular boundary points. We sketch the general situation in Fig. 1. In principle, by choosing a different gauge fixing in the neighborhood of these points one could resolve the singularity. If singular boundary points would not exist, all that would have been required is to complement the Hamiltonian in the Coulomb gauge with the appropriate boundary conditions in field space. Since the boundary identifications are by gauge transformations the boundary condition on the wave functionals is simply that they are identical under the boundary identifications, possibly up to a phase in case the gauge transformation is homotopically non-trivial.

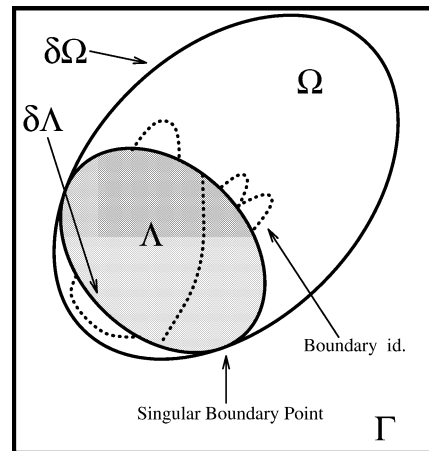


Fig. 1. Sketch of the fundamental (shaded) and Gribov regions, embedded in the space of transverse gauge fields (Γ). The dotted lines indicate boundary identifications.

Singular boundary points are to be expected [7]. Generically, at singular boundary points the norm functional undergoes a bifurcation moving from inside to outside the fundamental (and Gribov) region. The absolute minimum turns into a saddle point and two local minima appear. These are necessarily gauge copies of each other. The gauge transformation is homotopically trivial as it reduces to the identity at the bifurcation point, evolving continuously from there on.

The necessity to restrict to the fundamental domain, a subset of the transverse gauge fields, introduces a non-local procedure in field space. This cannot be avoided since it reflects the non-trivial topology of this space. We stress again that its topology and geometry are scale independent. Homotopical non-trivial gauge transformations are in one to one correspondence with non-contractible loops in field space, which give rise to conserved quantum numbers. The quantum numbers are like the Bloch momenta in a periodic potential and label representations of the homotopy group of gauge transformations. On the fundamental domain the non-contractible loops arise through identifications of boundary points. Although slightly more hidden, the fundamental domain will, therefore, contain all the information relevant for the topological quantum numbers. Sufficient knowledge of the boundary identifications will allow for an efficient and natural projection on the various superselection sectors. Typically we in-

tegrate out the high-energy modes, being left with the low-energy modes whose dynamics is determined by an effective Hamiltonian defined on the fundamental domain (restricted to these low-energy modes). In this it is assumed that the contributions of the high-energy modes can be dealt with perturbatively, generating the running coupling and the effective interactions of the low-energy modes.

With the boundary identifications implemented, and the fact that by construction $\|A\|_{\min}^2$ respects these boundary identifications,

$$\langle 0 \| A \|_{\min}^2 | 0 \rangle \equiv \int_A \mu(A) \mathcal{D}A \Psi_0^*(A) \| A \|_{\min}^2 \Psi_0(A) \quad (14)$$

is in principle well-defined, and could form the basis for establishing the existence of a non-perturbative dimension two condensate. Here $\Psi_0(A)$ is the ground-state wave functional, and $\mu(A)$ is the appropriate measure on field space [4,9], the integral assumed to be confined to the fundamental domain A of the transverse gauge fields.

5. Small volume results

In a small volume with periodic boundary conditions the running coupling is small and one can use perturbation theory. To lowest order the wave functional is simply a product of harmonic oscillators for each of the field modes. The zero-momentum modes, however, need to be treated separately since the potential in this sector is quartic. In computing $\langle 0 \| A \|_{\min}^2 | 0 \rangle$ we use for these zero-momentum modes the ground-state wave function belonging to the Hamiltonian of Eq. (9), truncated to the zero-momentum modes,

$$H_0 = -\frac{g^2}{2L} \left(\frac{\partial}{\partial c_j^a} \right)^2 - \frac{1}{2g^2 L} \text{tr}([c_j, c_k]^2), \quad (15)$$

where we defined $L^{-3} \int d^3x A_j^a(\vec{x}) = c_j^a/L$ and $c_j = c_j^a T^a$. A simple rescaling of the fields with a factor $g^{2/3}$ shows that the energies of this truncated Hamiltonian are proportional to $g^{2/3}/L$ and that the zero-momentum contribution to $\langle A^2 \rangle \equiv L^{-3} \langle 0 \| A \|_{\min}^2 | 0 \rangle$ is proportional to $g^{4/3}/L^2$.

It is in the direction of the zero-momentum modes that the wave functional will first reach the boundary of the fundamental domain for increasing coupling

(due to an increase in the volume). There is no classical potential along the direction of the Abelian zero-momentum modes (for which the commutator $[c_i, c_j]$ vanishes) and the boundary of the fundamental domain in these Abelian zero-momentum components can be shown [8] to occur at $(c_i^a)^2 = \pi^2$. Lüscher [12] has derived an effective Hamiltonian for the zero-momentum modes that incorporates the higher order corrections due to the interactions with the non-zero momentum modes, using so-called Bloch perturbation theory [13]. Amongst other things, this turns the bare coupling constant into a running coupling $g(L)$. It is beyond the scope of this Letter to describe the details of this calculation, but we remark that an efficient way to compute $\langle 0 \| A \|_{\min}^2 | 0 \rangle$ is by adding $\lambda(A_j^a(\vec{x}))^2$ to the Hamiltonian density of Eq. (9). Here $A_j^a(\vec{x})$ is assumed to be transverse and to lie within the fundamental domain. We will consider volumes where the restriction to the fundamental domain is felt for the zero-momentum modes only, such that we can integrate out the non-zero momentum modes perturbatively. The resulting effective Hamiltonian will contain a term depending on λ , but independent of the zero-momentum gauge field. This term will be dropped, as its value and its derivative with respect to λ , at $\lambda = 0$, can be interpreted as the perturbative contribution to the vacuum energy E_0 , respectively $\langle 0 \| A \|_{\min}^2 | 0 \rangle$. Keeping only the terms in the effective Hamiltonian that depend on the zero-momentum gauge field (and on λ) we may calculate the groundstate energy as a function of λ . Its derivative at $\lambda = 0$ gives the finite volume non-perturbative contribution to $\langle 0 \| A \|_{\min}^2 | 0 \rangle$.

Adding a mass term in the zero-momentum sector seems to remove the quartic nature of the potential. However, as rescaling the zero-momentum component of the gauge field with $g^{2/3}$ reveals, the term quadratic in the zero-momentum gauge field is proportional to $g^{4/3}$, and hence of lower order. As a matter of fact, the quantum corrections induce a term of this order in the effective Hamiltonian [12]. In Fig. 2 we illustrate for $SU(2)$ the lowest order result, with and without incorporating the boundary conditions, based on Ref. [14] (in terms of the terminology introduced there, the lowest order result is type IIIA with, and type IIIC without incorporating the boundary conditions; the computer code used here is essentially the one developed for that paper). In this figure we have shown the result up to $g = 2.9$, which corresponds

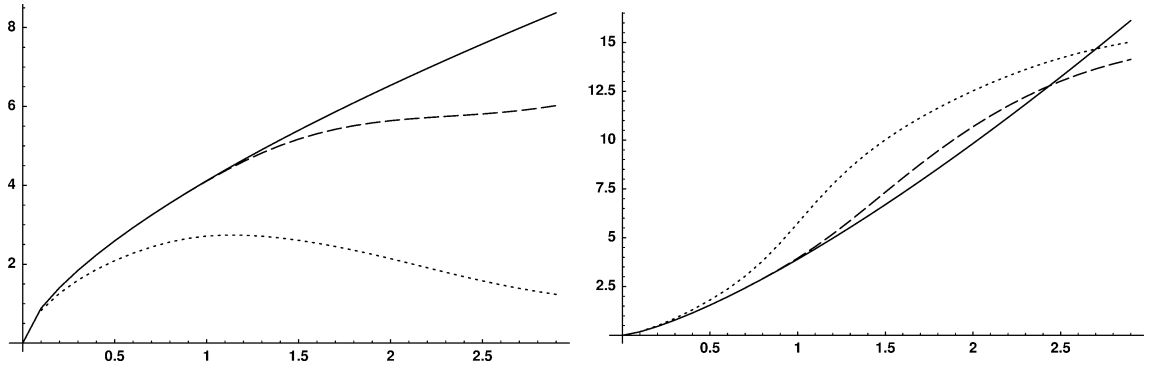


Fig. 2. On the left is shown LE_0 and on the right $L^2\langle A^2 \rangle$ for $SU(2)$, after subtracting the perturbative contribution as discussed in the text, both as a function of $g(L)$. The full lines give the lowest order result, $4.11672g^{2/3}(L)$, respectively $3.89775g^{4/3}(L)$. The dashed lines include the effect of the boundary conditions, for the lowest order effective Hamiltonian. The dotted lines include higher order contributions to the effective Hamiltonian (see the discussion in the text).

roughly to a volume of a cubic fermi. For larger volumes the wave functional will have spread in other directions as well, such that the perturbative approximation for these non-zero momentum modes no longer holds. Here we only wish to illustrate the influence boundary conditions in field space can have. It should, however, be understood that where a deviation starts to occur, the coupling is already sizable and a fully self-consistent calculation requires us to also include the higher order contributions to the effective Hamiltonian. To give a flavor of the magnitude of these corrections we show with the dotted curves the result obtained when ignoring the dependence of the groundstate wave function on λ .

6. Conclusion

We have shown how we can in principle define $\langle A^2 \rangle$ as $L^{-3}\langle 0||A||_{\min}^2|0\rangle$ in a Hamiltonian formalism, which restricts A to the fundamental domain. No two gauge fields in the interior of the domain are gauge equivalent, but gauge fields on the boundary in general do have gauge copies, also on the boundary. Crucial for defining $\langle 0||A||_{\min}^2|0\rangle$ is that $\|A\|_{\min}$ takes on the same value for these gauge equivalent gauge fields (as is intrinsic to the definition of the boundary of the fundamental domain).

To lowest order, in a small volume, this expectation value is generated in the usual way by adding the appropriate source to the Hamiltonian. The resulting

effective Hamiltonian is specified in terms of the gauge field components that will feel the boundary of the fundamental domain, which in a cubic finite volume smaller than about 0.75 cubic fermi means an effective Hamiltonian in terms of the zero-momentum modes. Although in this domain there is near perfect agreement with the low-lying spectrum obtained from lattice gauge theory, it is known that larger volumes are required to get close to the infinite volume limit. Nevertheless, we have shown a framework in which we can make sense of this quantity, even though its applicability is for technical reasons still limited to a finite volume.

Acknowledgements

P.v.B. is grateful for the hospitality extended to him at the MPI in Munich.

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