The group theory of twist eating solutions

by Bert van Geemen[1] and Pierre van Baal[2]

[1] *Department of Mathematics, Budapestlaan 6, P.O. Box 80.010, 3508 TA Utrecht, the Netherlands*
[2] *Institute for Theoretical Physics, State University of New York at Stony Brook, Long Island, New York 11794, USA*

Communicated by Prof. G. 't Hooft at the meeting of September 30, 1985

ABSTRACT

In this note we show the relation between solutions to the equation $[\Omega_\mu, \Omega_\nu] = \exp(2\pi i n_{\mu\nu}/N)I$ and the representations of the Heisenberg group, where $\mu = 1,2,\ldots,2g$ and $\Omega_\nu \in SU(N)$. We construct all its irreducible representations and whence all solutions to the above equation for arbitrary g. We give a criterium for existence and uniqueness of solutions.

1. THE SETTING OF TWIST EATING SOLUTIONS

Twisted gauge fields on the hypertorus, both in the continuum and on the lattice posed an interesting mathematical problem, namely finding $SU(N)$ matrices Ω_μ (called twist eating solutions), such that:

$$(1)\qquad \begin{cases} [\Omega_\mu, \Omega_\nu] = \Omega_\mu \Omega_\nu \Omega_\mu^{-1} \Omega_\nu^{-1} \\ \qquad\qquad = \exp(2\pi i n_{\mu\nu}/N) \cdot I \end{cases}$$

n is called the twist tensor, it is skew symmetric with integer entries mod N. The index μ runs from 1 to $2g$ (the dimension of space-time; odd dimensions need not be considered separately). For details and further references see [1, 2] where the full solution of this problem for $g \leq 2$ was found.

By means of a $Sl(2g, \mathbb{Z})$ transformation X, we can always transform n to its standard form:

$$n^{(0)} = \begin{pmatrix} & \Theta & e_1 & & \\ & & & \ddots & \\ -e_1 & & & & e_g \\ & \ddots & & & \\ & & -e_g & \Theta & \end{pmatrix} \tag{2}$$

where[1] $e_1|e_2|\ldots|e_g$ and $n = X^t n^{(0)} X$. If $[U_\mu, U_\nu] = \exp(2\pi i n^{(0)}_{\mu\nu}/N)I$, then equation (1) is solved by

$$\Omega_\mu = \prod_\nu U_\nu^{X_{\nu\mu}}. \tag{3}$$

From now on we assume n to be in the form (2).

This form is not unique since we can add a multiple of N to $n_{\mu\nu}$. However since eq. (3) can be inverted [2], the specific choice of $n^{(0)}$ is irrelevant.

Previously we established that solutions exist iff[2] $Pf(n/N)N$ is an integer, for $g = 1, 2$ (where Pf is the Pfaffian which is a squareroot of the determinant of an even dimensional skew symmetric matrix; $Pf(n^{(0)}) = -\prod_{j=1}^{g}(-e_j)$). In this note we will show that for arbitrary g, $Pf(n/N)N \in \mathbb{Z}$ is a necessary but *not a sufficient* condition for existence of solutions to eq. (1), consistent with the remark in [1] that $Pf(n) = 0 \bmod N$ is not a sufficient condition.

It was also shown for $g \leq 2$ that the solution to equation (1) is unique up to a similarity transformation and multiplication with an element of the centre of $SU(N)$ iff[2]:

$$\text{g.c.d.}\left(n_{\mu\nu}, Pf\left(\frac{n}{N}\right)N, N\right) = 1. \tag{4}$$

For $g > 2$ this is a sufficient, but *not a necessary* condition. We will show that uniqueness up to a similarity transformation means that the matrices U generate an irreducible representation of the Heisenberg group, in which case there are $N^{2(g-1)}$ $SU(N)$-inequivalent solutions.

It is clear that the group G generated by U has the property that its commutator is in the centre of the group G (G is therefore nilpotent). This is typical of the so-called Heisenberg group. (Indeed also in physics the Heisenberg commutation relation between coordinates and their canonical momenta satisfy the same property).

In the next section we will describe the relevant Heisenberg group and its Schrödinger representation. In section 3 all representations of the Heisenberg group are classified. In section 4 these results are used to construct all solutions to (1) and give the approporiate criteria for existence and uniqueness, based on n.

---

[1] For integer p and q the symbol $p|q$ means that p divides q.

[2] iff = if and only if, g.c.d. = greatest common divisor.

In general one can describe the Heisenberg group by the following properties. Let K be an (additive abelian) group, $\mathbb{C}^* = \mathbb{C} - \{0\}$ the multiplicative group of complex numbers and K^* the dual of K, i.e. K^* is the (additive) group of homomorphisms $f: K \to \mathbb{C}^*$. We will denote its elements by x^*. The Heisenberg group is given by $H = \mathbb{C}^* \times K \times K^*$, with a product defined by:

$$(5) \qquad \begin{cases} H \times H \to H: (t, x, x^*) \cdot (s, y, y^*) \\ \qquad = (tsy^*(x), x+y, x^*+y^*). \end{cases}$$

For completeness let us mention the following properties

$$(6) \qquad \begin{cases} x^*(a+b) & = x^*(a) \cdot x^*(b) \\ (x^*+y^*)(a) & = x^*(a) \cdot y^*(a) \\ (t, x, y^*)^{-1} & = (t^{-1}y^*(x), -x, -y^*) \\ [(t, x, y^*), (s, u, v^*)] & = (v^*(x) \cdot y^*(u)^{-1}, 0, 0). \end{cases}$$

In principle we can restrict ourselves to the subgroup

$$\{t \in \mathbb{C}^* | \exists x \in K,\ y^* \in K^*: t = y^*(x)\} \text{ of } \mathbb{C}^*.$$

When K is finite this makes H a finite group, which is convenient for finding all its representations. In fact when K is finite we must have that $y^*(x)$ is a root of unity for all x and y^*.

We will now introduce the Heisenberg group $H(\delta)$, of type δ, where:

$$(7) \qquad \begin{cases} \delta = (d_1, d_2, \ldots, d_g) \\ d_i \in \mathbb{N}, d_1 | d_2 \ldots | d_g. \end{cases}$$

The length or norm of δ is given by:

$$|\delta| = d_g.$$

To δ we associate the group $K(\delta)$:

$$(8) \qquad K(\delta) = Z_{d_1} \times Z_{d_2} \times \ldots Z_{d_g}$$

where Z_n stands for $\mathbb{Z}/n\mathbb{Z}$, being the additive group of integers modulo n. To distinguish this from the unimodular group of the n-th roots of unity the latter will be denoted by μ_n, which is a multiplicative group:

$$(9) \qquad \begin{cases} \mu_n = \{1, e^{2\pi i/n}, e^{4\pi i/n}, \ldots, e^{2\pi i(n-1)/n}\} \\ \quad = \{t \in \mathbb{C}^* | t^n = 1\}. \end{cases}$$

The centre of $SU(N)$ is also denoted by μ_N. We define $H(\delta)$ by:

$$H(\delta)=\mu_{|\delta|}\times K(\delta)\times K(\delta)^*.$$

Let $q:\mathbb{Z}^g\to K(\delta)$ be the canonical projection.

Define an isomorphism $K(\delta)\to K(\delta)^*, x\to *(x)$ by:

$$(10)\qquad *(q(x))(q(y))=\prod_{j=1}^{g}\exp\left(\frac{2\pi i x_j y_j}{d_j}\right)$$

$$\left(\text{so } [(t,q(x),*(q(y)),(s,q(u),*(q(v)))]=\left(\exp\left(2\pi i\sum_k \frac{v_k x_k-y_k u_k}{d_k}\right),0,0\right)\right).$$

We will next construct the canonical representation of $H(\delta)$, called the Schrödinger-representation, which is denoted by σ_δ. The representation space will be the $\mathbb{C}$-vector space of functions $f:K(\delta)\to\mathbb{C}$. Define an action of $H(\delta)$ on this vector space of dimension $N=\prod_{j=1}^{g} d_j$ by:

$$(11)\qquad \{[\sigma_\delta(t,x,y^*)](f)\}(z)=ty^*(z)f(x+z).$$

Let us construct explicitely the N-dimensional unitary matrix representation of σ_δ. A $\mathbb{C}$-basis of Func $(K(\delta)\to\mathbb{C})$ is given by

$$(12)\qquad \begin{cases} f_a(x)=1 & x=a, \quad x,a\in K(\delta)\\ \quad\;\; =0 & x\neq a\end{cases}$$

or equivalently

$$(13)\qquad \begin{cases} f_{q(a)}(q(x))=1 & x_j=a_j \bmod d_j\\ \qquad\qquad\;\; =0 & x_j\neq a_j \bmod d_j\end{cases}\quad (\text{all } j).$$

In an obvious notation: $f_a(x)=\delta_{a,x}$.

It is easy to check that the matrix $\sigma_\delta(t,x,y^*)_{ab}$ is defined by:

$$(14)\qquad [\sigma_\delta(t,x,y^*)](f_b)=\sum_{a\in K(\delta)}\sigma_\delta(t,x,y^*)_{ab}f_a$$

i.e.

$$(15)\qquad \begin{cases} [\sigma_\delta((t,x,y^*)\cdot(s,u,v^*))](f_b)=\\ \displaystyle\sum_{a,c\in K(\delta)}[\sigma_\delta(t,x,y^*)]_{c,a}[\sigma_\delta(s,u,v^*)]_{a,b}f_c.\end{cases}$$

To write down the matrices we use:

$$(16)\qquad (t,q(x),*(q(y)))=(t,0,0)\cdot\prod_{j=1}^{g}(1,0,u_j^*)^{y_j}\cdot\prod_{j=1}^{g}(1,u_j,0)^{x_j}$$

with

$$(17)\qquad (u_j)_k=\delta_{j,k}.$$

If we introduce:

(18) $U_{g-j+1}=\hat{U}_j=\sigma_\delta(1,0,u_j^*),\quad U_{2g-j+1}=\hat{U}_{g+j}=\sigma_\delta(1,u_j,0)$

we have (from now on we ignore the difference between x and $q(x)$):

(19) $$\sigma_\delta(t,x,y)=t\prod_{j=1}^{g}\hat{U}_j^{y_j}\prod_{j=1}^{g}\hat{U}_{g+j}^{x_j}.$$

Explicitly we have:

(20) $$\begin{cases}(U_{g-k+1})_{a,b}=\exp\left(\dfrac{2\pi i a_k}{d_k}\right)\delta_{a,b}\\(U_{2g-k+1})_{a,b}=\delta_{a+u_k,b}.\end{cases}$$

Or as tensor products we can write:

(21) $$\begin{cases}U_{g-k+1}=1_{d_1}\otimes\ldots\otimes Q_{d_k}\otimes\ldots\otimes 1_{d_g}\\U_{2g-k+1}=1_{d_1}\otimes\ldots\otimes P_{d_k}\otimes\ldots\otimes 1_{d_g}\end{cases}$$

where 1_n is the n-dimensional identity and Q_n, P_n are the twist matrices satisfying

(22) $[P_n,Q_n]=e^{2\pi i/n}\cdot 1_n$

with

$$Q_n=\operatorname{diag}(1,e^{2\pi i/n},\ldots,e^{2\pi i(n-1)/n})$$

(23) $$P_n=\begin{pmatrix}0&1&&0\\\vdots&0&\ddots&\\\vdots&&\ddots&1\\1&\ldots&\ldots&0\end{pmatrix}.$$

It is an easy exercise to show that this representation is irreducible. Furthermore we have that U_μ satisfy eq. (2) with:

(24) $$e_j=-\frac{N}{d_{g-j+1}},\quad N=\prod_{K=1}^{g}d_k.$$

Note that $Pf(n/N)=-N^{-1}$; therewith we verified for this particular representation: $Pf(n/N)N$ is integer and g.c.d. $(n_{\mu\nu},N,Pf(n/N)N)=1$.

3. ALL REPRESENTATIONS OF THE HEISENBERG GROUP $H(\delta)$

In order to find all representations, we can invoke the well known result [3] due to Frobenius for a finite group H:

(25) $$\sum_\varrho(\dim\varrho)^2=0(H)$$

where the summation runs over the irreducible representations ϱ and $0(H)$ is the order (number of elements) of H. We have:

$$(26)\qquad 0(H(\delta)) = |\delta|\Big(\prod_{j=1}^{g} d_j\Big)^2.$$

Denote by $C(\delta)$ the centre of $H(\delta)$

$$(27)\qquad C(\delta) = \{(t,0,0)\,|\,t\in\mu_{|\delta|}\}$$

and let ϱ be some irreducible representation of $H(\delta)$ on a vector space V ($\dim\varrho := \dim V$). Then by Schur's lemma:

$$(28)\qquad \varrho(c) = \lambda_\varrho(c)I, \qquad \forall c\in C(\delta)$$

with I the identity on V and $\lambda_\varrho(c)\in\mathbb{C}$.

Obviously $\lambda_\varrho : C(\delta)\to\mathbb{C}$ is a homomorphism which we will call the central character (of ϱ). Note that if two irreducible representations ϱ and ϱ' satisfy $\lambda_\varrho\neq\lambda_{\varrho'}$, they can not be equivalent. Since $C(\delta)$ is cyclic of order $|\delta|$ we must have:

$$(29)\qquad \lambda_\varrho((t,0,0)) = t^m, \qquad 0\le m\le|\delta|-1.$$

The following, so-called twisted Schrödinger representation has this central character:

$$(30)\qquad \{[\sigma_\delta(m)(t,x,y^*)](f)\}(z) = [ty^*(z)]^m f(x+z).$$

Following the same steps as in the previous section we find the matrix representation of $\sigma_\delta(m)$ by:

$$(31)\qquad \begin{cases} U_{g-k+1} = \sigma_\delta(m)(1,0,u_k^*) = 1_{d_1}\otimes\ldots\otimes Q_{d_k}^m\otimes\ldots\otimes 1_{d_g} \\ U_{2g-k+1} = \sigma_\delta(m)(1,u_k,0) = 1_{d_1}\otimes\ldots\otimes P_{d_k}\otimes\ldots\otimes 1_{d_g}. \end{cases}$$

An invariant subspace W is found by considering each tensor component separately. For the k-th component we have ($\boldsymbol{z} = (1,1,1\ldots1)\in\mathbb{C}^{d_k}$)

$$(32)\qquad W_k = \langle \boldsymbol{z}, Q_{d_k}^m\boldsymbol{z}, Q_{d_k}^{2m}\boldsymbol{z},\ldots,Q_{d_k}^{m(d_k-1)}\boldsymbol{z}\rangle.$$

Using the explicit diagonal form of Q_{d_k} in eq. (23) we find[3]:

$$(33)\qquad \dim W_k = d_k/\text{g.c.d. }(m,d_k).$$

On the other hand, when g.c.d. $(m,d_k)=1$ the eigenvalues of $Q_{d_k}^m$ are a permutation of those of Q_{d_k}, which implies that there is no non-trivial invariant subspace.

Let us introduce the notation:

$$(34)\qquad \begin{cases} \pi = (p_1,p_2,\ldots,p_g), & p_k = \text{g.c.d. }(m,d_k) \\ \gamma = (c_1,c_2,\ldots,c_g), & c_k = d_k/p_k. \end{cases}$$

[3] For $m=0$ we define g.c.d. $(m,d_k)=d_k$.

From eq. (7) we have that $p_g = 1$ implies that all other p_k equal 1. We conclude: $\sigma_\delta(m)$ is irreducible iff m and $|\delta|$ are relatively prime.

Assume now that ϱ is an irreducible representation of $H(\delta)$ with central character $\lambda_\varrho(t) = t^m$ and with g.c.d. $(m, |\delta| = d_g) \neq 1$. Define π, γ as above and

(35) $$C(\delta, m) = (\mu_{|\delta|}, \gamma K(\delta), (\gamma K(\delta))^*),$$

where we used the shorthand notation:

(36) $$\gamma K(\delta) = c_1 Z_{d_1} \times c_2 Z_{d_2} \times \ldots \times c_g Z_{d_g}.$$

Note that elements of $\varrho(C(\delta, m))$ commute with $\varrho(H(\delta))$, hence by Schur's lemma $\varrho(C(\delta, m))$ consists of scalar multiples of the identity. Therefore ϱ restricted to $C(\delta, m)$ is given by ($a, b \in K(\pi)$ and $\gamma \cdot x := (c_1 x_1, \ldots, c_g x_g) \in K(\delta)$):

(37) $$\varrho(t, \gamma \cdot x, \gamma \cdot y^*) = t^m \chi_{a,b}(x, y) I$$

with

(38) $$\chi_{a,b}(x, y) = \exp\left(2\pi i \sum_k \frac{x_k a_k + y_k b_k}{p_k}\right).$$

Clearly two irreducible representations with different m, a, b would be inequivalent.

Let us now consider the case where $a = b = 0$. Then ϱ is trivial on the subgroup $C_0(\delta, m)$ of $H(\delta)$:

(39) $$C_0(\delta, m) = ((\mu_{|\delta|})^{|\gamma|}, \gamma K(\delta), (\gamma K(\delta))^*).$$

Therefore ϱ factors over $C_0(\delta, m)$:

(40) $$\begin{array}{ccc} H(\delta) & \xrightarrow{\ \varrho\ } & V \\ \downarrow & \nearrow_{\bar{\varrho}} & \\ H(\delta)/C_0(\delta, m) & & \end{array}$$

$\bar{\varrho}$ is a faithfull representation of $H(\delta)/C_0(\delta, m)$.

On the other hand $H(\delta)/C_0(\delta, m)$ is isomorphic to $H(\gamma)$, which is most easily seen by defining a homomorphism $\phi^\delta_\gamma : H(\delta) \to H(\gamma)$ whose kernel coincides with $C_0(\delta, m)$:

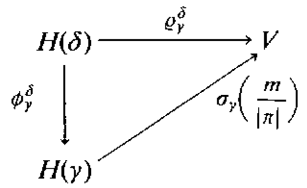
(41) $$\phi^\delta_\gamma : (t, x, y^*) \to (t^{|\pi|}, \psi(x), \psi^*(y^*))$$

with

(42) $$\begin{cases} \bar{x}_k & = x_k \bmod c_k \\ \psi_k(x) & = (p_k^{-1} |\pi|) \bar{x}_k \\ \psi^*(y^*)(\bar{x}) & = \exp\left(2\pi i \sum_k \dfrac{y_k x_k}{p_k^{-1} d_k}\right). \end{cases}$$

(Note that: i) $\psi^*(y^*)=\bar{y}^*$ in an obvious way, however one should be careful with this notation when one identifies $q(x)$ with x, ii) $\psi: K(\delta)\to K(\gamma)$ is surjective because $p_i^{-1}|\pi|$ and c_i are relatively prime iii) $\psi(\gamma\cdot x)=0$ and $\psi^*(\gamma\cdot y^*)=0$.)

It is now obvious that an irreducible representation with central character t^m is given by: $\varrho_\gamma^\delta=\sigma_\gamma(m/|\pi|)\circ\phi_\gamma^\delta$

(43) 

Explicitly (with $f\in \mathrm{Func}(K(\gamma),\mathbb{C})$, $\bar{z}\in K(\gamma)$):

(44) $$(\varrho(t,x,y^*)\bar{f})(\bar{z}))=(t^{|\pi|}\psi^*(y^*)(\bar{z}))^{m/|\pi|}\bar{f}(\psi(x)+\bar{z}).$$

Finally for $a,b\neq 0$ we can extend $\chi_{a,b}$ uniquely to a 1-dimensional representation of $H(\delta)$, with (necessarily) $m=0$:

(45) $$\tilde{\chi}_{a,b}(t,x,y)=\exp\left(2\pi i\sum_k\frac{x_ka_k+y_kb_k}{d_k}\right).$$

The product representation:

(46) $$\varrho=\tilde{\chi}_{a,b}\cdot\varrho_\gamma^\delta$$

is therefore an irreducible representation for each m, a and b.

We have:

(47) $$\begin{cases}\displaystyle\sum_{m=0}^{|\delta|-1}\sum_{a,b\in K(\pi)}\dim(\tilde{\chi}_{a,b}\cdot\varrho_\gamma^\delta)^2\\[2ex] \displaystyle=\sum_{m=0}^{|\delta|-1}(\prod_{k=1}^{g}p_k)^2(\prod_{k=1}^{g}c_k)^2=|\delta|\prod_{j=1}^{g}d_j^2.\end{cases}$$

Using (25) and (26) we see that we have found all irreducible representations. They are in essence all twisted Schrödinger representations.

4. BACK TO THE TWIST

Let us first write down the matrices U_μ for ϱ_γ^δ:

(48) $$\begin{cases}U_{g-k+1}=\varrho_\gamma^\delta(1,0,u_k^*)=1_{c_1}\otimes\ldots\otimes Q_{c_k}^{m/|\pi|}\otimes\ldots\otimes 1_{c_g}\\ U_{2g-k+1}=\varrho_\gamma^\delta(1,u_k,0)=1_{c_1}\otimes\ldots\otimes P_{c_k}^{|\pi|/p_k}\otimes\ldots\otimes 1_{c_g}\end{cases}$$

which satisfy eq. (2) with:

(49) $$e_{g-k+1}=-N\frac{m/p_k}{c_k}=-\frac{Nm}{d_k},\qquad N=\prod_{k=1}^{g}c_k:=N_{irr}.$$

Furthermore we have:

$$(50) \qquad Pf\left(\frac{n}{N}\right)N = -\prod_{k=1}^{g}\left(\frac{m}{p_k}\right) \in \mathbb{Z}.$$

Clearly $m_k \equiv m/p_k$ is relatively prime to c_k. Since $c_i|c_k$ for $i<k, m_k$ is also relative prime to c_i.

For $g \leq 2$ this is easily seen to imply g.c.d. $(n_{\mu\nu}, NPf(n/N), N) = 1$. However, the following example, due to Coste, shows that it is not a necessary condition for irreducibility. Take:

$$(51) \qquad \begin{cases} g=3,\ N=2^2\cdot 7^3 \\ e_3=4e_2=4e_1=2^3\cdot 3\cdot 7^2. \end{cases}$$

For this, one explicitly verifies that:

$$(52) \qquad \text{g.c.d.}\left(n_{\mu\nu}, N\cdot Pf\left(\frac{n}{N}\right), N\right)=2\neq 1,$$

but that it nevertheless admits a solution which is based on the irreducible representation $\sigma_\delta(m)$ with:

$$(53) \qquad \delta=(7,28,28), \qquad m=6.$$

To show that $N\cdot Pf(n/N)\in\mathbb{Z}$ is not a sufficient condition[4] we give the next example, also provided by Coste:

$$(54) \qquad \begin{cases} g=3, N=2^2\cdot 3^6 \\ e_3=2^4e_2=2^4e_1=2^4 3^4. \end{cases}$$

This yields $Pf(n/N)N=1$, but it cannot allow for a twist eating solution. One way to see this is to use the well known result, that

$$(55) \qquad U(k)=\prod_{\mu=1}^{6} U_\mu^{k_\mu}$$

are independent $N\times N$ matrices for $0\leq k_\mu<N_\mu$ if a solution does exist. Where:

$$(56) \qquad N_i=N_{g+i}=c_{g-i+1}.$$

This is an easy generalization of the well known result for $g=1$ and 2, see e.g. [2].

Therefore we have at least $\prod_\mu N_\mu=N_{irr}^2$ independent $N\times N$ matrices. There can be no more than N^2, so that necessarily

$$(57) \qquad N_{irr}\leq N.$$

---

[4] Integrality of $Pf(n/N)N$ in general depends on the choice of n; i.e. it depends on its mod N freedom.

For the case of eq. (54) this bound is easily seen to be violated, hence $Pf(n/N)N \in \mathbb{Z}$ is not sufficient for the existence of solutions to eq. (1).

Since for given e_j and N, c_j and m_j are fixed, one can always find d_j and m such that eq. (7) is satisfied. The simplest choice for m is the smallest common multiple of all m_i, which equals m_1 (see below eq. (50)).

Therefore if N is a multiple of N_{irr} we can write down the following solution:

$$U_\mu = \Lambda_\mu \times U_\mu^{irr}, \tag{58}$$

where: $\Lambda_\mu = \mathrm{diag}\,(\lambda_\mu^{(1)}, \ldots, \lambda_\mu^{(N/N_{irr})}) \in U(1)^{N/N_{irr}}$ and $U_\mu^{irr} \in U(N_{irr})$ coming from $\sigma_\delta(m)$ (see eq. (48)).

We claim that up to a similarity transformation this is the only possible type of solution. Suppose that U_μ is a solution and define:

$$\omega_\mu = U_\mu^{N_\mu}, \tag{59}$$

with N_μ defined in (56) satisfying $Z_{\mu\nu}^{N_\mu} = 1$. This implies that the ω_μ can be simultaneously diagonalized, with U_μ block-diagonal:

$$[\omega_\mu, \omega_\nu] = [\omega_\mu, U_\nu] = 1\,. \tag{60}$$

Working in this diagonal gauge one easily verifies that:

$$\hat{U}_\mu = \Lambda_\mu^{-1} U_\mu, \qquad \Lambda_\mu^g = \omega_\mu \tag{61}$$

satisfy the same equation as U_μ, but such that $\hat{\omega}_\mu = \hat{U}_\mu^{N_\mu} = 1$. Consequently, the $\hat{U}_\mu$ give a representation of the Heisenberg group. With the help of the previous two sections there is no other possibility for $\hat{U}_\mu$ then to be the direct sum of irreducible representation of dimension N_{irr}. And hence N has to be a multiple of N_{irr} and U_μ is of the form of eq. (58). The dimension of the solution manifold to equation (1) up to a similarity transformation is $(N/N_{irr} - 1)$. Given a solution we therefore find[5]:

$$Pf\left(\frac{n}{N}\right)N = -\left(\prod_{k=1}^{g} m_k\right) \cdot \frac{N}{N_{irr}} \in \mathbb{Z} \tag{62}$$

and

$$e_{g-i+1} = \frac{mN}{d_i} = \frac{N}{N_{irr}}\, m_i \prod_{k \neq i} c_k, \tag{63}$$

whence:

$$(N/N_{irr}) | \text{g.c.d.} \left(n_{\mu\nu}, Pf\left(\frac{n}{N}\right)N, N\right). \tag{64}$$

One easily verifies that for $g = 1$ and 2, g.c.d. $(n_{\mu\nu}, Pf(n/N)N, N) = N/N_{irr}$ and the dimension of the solution manifold confirms with previous results [1].

---

[5] For $g = 1$ and 2, g.c.d. $(N_{irr}, \prod_{k=1}^{g} m_k) = 1$, so that $Pf(n/N)N \in \mathbb{Z}$ implies that N is a multiple of N_{irr}.

We also see that g.c.d. $(n_{\mu\nu}, Pf(n/N)N, N)=1$ implies $N=N_{irr}$, which implies uniqueness of the solution. Since N_{irr} fixes the dimensionality of the solution manifold, we indirectly see that in the case that N is a multiple of N_{irr} for a specific choice of n, N_{irr} does not depend on this choice (because eq. (3) is invertable [2].)

In conclusion, twist eating solutions exist iff N is a multiple of N_{irr} (for which $Pf(n/N)N \in \mathbb{Z}$ is only a necessary condition). A solution is unique up to a gauge and Z_N-factors iff $N=N_{irr}$ iff it corresponds to an irreducible representation (for which g.c.d. $(n_{\mu\nu}, Pf(n/N)N, N)=1$ is only a sufficient condition), in that case (48) gives an explicit solution for eq. (2), which through a simple rescaling by a phasefactor can be chosen in $SU(N)$

$$\left(\text{replace } Q_n \text{ by } \hat{Q}_n \equiv \exp\left(-\frac{\pi i(n-1)}{n}\right)\cdot Q_n \text{ and similar for } P_n\right).$$

Multiplication with a phasefactor is indeed the only freedom we have, which for $SU(N)$ reduces to multiplication with elements of the centre of $SU(N)$ (isomorphic to μ_N). Using the fact that

$$\text{g.c.d. } (m|\pi|^{-1}, c_k) = \text{g.c.d. } (|\pi|\, p_k^{-1}, c_k) = 1$$

we have that $\lambda \cdot \hat{Q}_{c_k}^{m/|\pi|}$ and $\lambda \cdot \hat{P}_{c_k}^{|\pi|/p_k}$ are equivalent to $\hat{Q}_{c_k}^{m/|\pi|}$ and $\hat{P}_{c_k}^{|\pi|/p_k}$ if $\lambda \in \mu_{c_k}$. Therefore all inequivalent solutions to equation (2) are given by (see (48)) $\lambda_k U_{g-k}$ and $\nu_k U_{2g-k}$ with $\lambda_k, \nu_k \in \mu_N/\mu_{c_k}$. Hence there are

$$\prod_{k=1}^{g} 0(\mu_N/\mu_{c_k})^2 = N^{2(g-1)}$$

inequivalent solutions.

5. CONCLUSIONS

In this note new results concern twist eating solutions for more than four dimensions; one might think of applications for TEK-models in the $d\to\infty$ limit, where d is the dimension of space-time. However our main motivation was to show the underlying structure of the Heisenberg group.

ACKNOWLEDGEMENTS

We would like to thank prof. T.A. Springer for a remark which made us see the light. B.v.G. is financially supported by the ZWO project on moduli and P.v.B. by the Dutch National Science Foundation FOM (ZWO). This work was partially supported by NSF grant PHY 81-09110 A-03. One of us (P.v.B.) is also very grateful to Antoine Coste for the counter examples to the $g=2$ criteria when extended to $g \geq 2$.

REFERENCES

1. Baal, P. van – Communications in Mathematical Physics, **92**, 1 (1983).
2. Baal, P. van – Twisted boundary conditions: A non-perturbative probe for pure non-Abelian gauge theories, thesis, Utrecht, juli 1984.
3. Serre, J.-P. – Représentation lineaires des groupes finis, Hermann, Paris, 1971 (2e éd.).